MATH 2300-04
PRACTICE FINAL – SOLUTIONS

VANDERBILT UNIVERSITY

Directions: This practice test should be used as a study guide, illustrating the concepts that will be emphasized in the first test. This does not mean that the actual test will be restricted to the content of the practice. Try to identify, from the questions below, the concepts and methods that you should master for the test. For each question in the practice test, study the ideas and techniques connected to the problem, even if they are not directly used in your solution.

Take this also as an opportunity to practice how you will write your solutions in the test. For this, write clearly, legibly, and in a logical fashion. Make precise statements (for instance, write an equal sign if two expressions are equal; say that one expression is a consequence of another when this is the case, etc.).

The final test will cover the following sections of the textbook: 15.3, 15.5, 15.6, 15.7, 15.8, 15.9, 16.1, 16.2, 16.3, 16.4, 16.5, 16.6, 16.7, 16.8, and 16.9 (15.4 will not be on the test). But in a sense, you should view it as cumulative in that you need the previous material to understand these sections.

Question 1. Let D be the solid bounded by $x = 2$, $y = 2$, $z = 0$, and $x + y - 2z = 2$. Express

$$\iiint_D f(x, y, z)\, dV$$

as an iterated integral.

Solution 1. The region is illustrated in figure 1. We can write

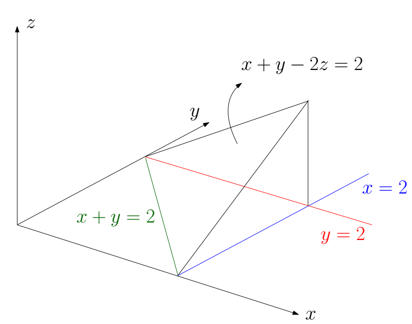

FIGURE 1. Region of question 1.

$$D = \{(x, y, z) \,|\, 0 \le x \le 2, 2 - x \le y \le 2,\, 0 \le z \le \frac{1}{2}(x + y - 2)\}.$$

Thus

$$\iiint\limits_D f(x,y,z)\,dV = \int_0^2 \int_{2-x}^2 \int_0^{\frac{1}{2}(x+y-2)} f(x,y,z)\,dzdydx.$$

Question 2. Let S be the paraboloid $z = x^2 + y^2$, $0 \leq z \leq 1$.

(a) Sketch the surface S.

(b) Write S as a parametric surface.

(c) Find an expression for the unit normal vector field to S. Orient it so that the normal points downward.

(d) Evaluate

$$\iint\limits_S \mathbf{F} \cdot d\mathbf{S},$$

where

$$\mathbf{F}(x,y,z) = -x\,\mathbf{i} - y\,\mathbf{j} + z\,\mathbf{k}.$$

Solution 2. (a) See figure 2.

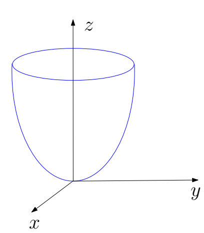

FIGURE 2. Surface of question 2.

(b) Parametrizing by in the variables x and y,

$$\mathbf{r}(x,y) = x\,\mathbf{i} + y\,\mathbf{j} + (x^2+y^2)\,\mathbf{k},\ x^2+y^2 \leq 1.$$

(c) Since the surface is given by a graph, we have that

$$-z_x\,\mathbf{i} - z_y\,\mathbf{j} + \mathbf{k} = -2x\,\mathbf{i} - 2y\,\mathbf{j} + \mathbf{k}$$

is a normal that points upward; thus,

$$\mathbf{n}(x,y) = \frac{2x\,\mathbf{i} + 2y\,\mathbf{j} - \mathbf{k}}{\sqrt{4x^2+4y^2+1}}.$$

(d) We have

$$\begin{aligned}
\iint_S \mathbf{F}\cdot d\mathbf{S}, &= \iint_S \mathbf{F}\cdot\mathbf{n}\,dS \\
&= \iint_D (-x\,\mathbf{i} - y\,\mathbf{j} + (x^2+y^2)\,\mathbf{k})\cdot(2x\,\mathbf{i} + 2y\,\mathbf{j} - \mathbf{k})\,dA \\
&= -3\iint_D (x^2+y^2)\,dA \\
&= -3\int_0^{2\pi}\int_0^1 r^2 r\,dr d\theta = -\frac{3}{2}\pi.
\end{aligned}$$

Question 3. Using a change of variables, evaluate the integral

$$\iint_D \frac{\sqrt{x^2+16y^2+8xy}}{2\sqrt{x^2+4y^2}}\,dA,$$

where D is the region bounded by $x^2+4y^2=4$, $x^2+4y^2=16$, $y-x-1=0$, and $y-x+2=0$, and $x\geq 0$.

Solution 3. The region is illustrated in figure 3.

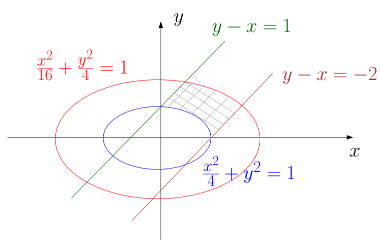

FIGURE 3. Region of question 3.

Set $u^2 = \frac{x^2}{4}+y^2$, $u\geq 0$, so that the inner ellipse (blue curve) corresponds to $u=1$ and the outer ellipse (red curve) to $u=2$. Set $v=y-x$, so that the two lines correspond to $v=1$ and $v=-2$.

Inverting the equations for $x=x(u,v)$ and $y=y(u,v)$ we find

$$x = \frac{2}{5}(-2v+\sqrt{5u^2-v^2}),$$

and

$$y = \frac{v}{5}+\frac{2}{5}\sqrt{5u^2-v^2}.$$

One can verify that these expressions give a well-defined transformation for $1\leq u\leq 2$ and $-2\leq v\leq 1$. The Jacobian of the change of variables is

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{2u}{\sqrt{5u^2-v^2}}.$$

On the other hand,

$$f(x(u,v),y(u,v)) = \frac{\sqrt{5u^2-v^2}}{2u},$$

thus

$$\iint_D \frac{\sqrt{x^2+16y^2+8xy}}{2\sqrt{x^2+4y^2}}\,dA = \int_{-2}^{1}\int_{1}^{2} \frac{\sqrt{5u^2-v^2}}{2u}\frac{2u}{\sqrt{5u^2-v^2}}\,du\,dv = 3.$$

Question 4. Let $\mathbf{F}$ be a vector field $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$. Let S_2 be the square with vertices at $(-2,-2)$, $(-2,2)$, $(2,2)$, and $(2,-2)$, and S_3 be the square with vertices at $(-3,-3)$, $(-3,3)$, $(3,3)$, and $(3,-3)$. Let $C_2 = \partial S_2$ and $C_3 = \partial S_3$, both oriented counter-clockwise. Assume that:

- The functions P and Q have continuous partial derivatives in the region $x^2+y^2 \geq 1$.
- $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$ in the region outside S_2.
- $\int_{C_2} \mathbf{F}\cdot d\mathbf{r} = 1$.

(a) Evaluate $\int_{C_3} \mathbf{F}\cdot d\mathbf{r}$.

(b) Is $\mathbf{F}$ a conservative vector field in the region $x^2+y^2 \geq 1$?

(c) What can you say about the functions P and Q in the region $x^2+y^2 \leq 1$?

Solution 4. The given curves are shown in figure 4.

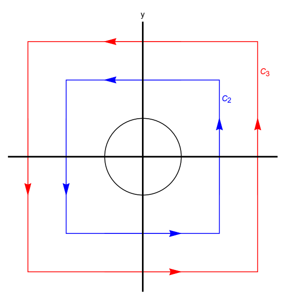

FIGURE 4. Curves of question 4.

(a) P and Q satisfy the assumption of Green's theorem in the region $x^2+y^2 \geq 1$. Let D be the region between the curves C_2 and C_3. With the orientation convention of Green's theorem, D is on the left of the curve C_3 when it is traversed counter-clockwise, and on the left of $-C_2$, since C_2 has been initially oriented counter-clockwise (and in such case D is not on the left of C_2). Thus, $\partial D = (-C_2)\cup C_3$. Invoking Green's theorem and using the given information,

$$\begin{aligned}\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = 0 &= \int_{\partial D} (P\,dx + Q\,dy) \\ &= \int_{-C_2} (P\,dx + Q\,dy) + \int_{C_3} (P\,dx + Q\,dy) \\ &= -\int_{C_2} \mathbf{F}\cdot d\mathbf{r} + \int_{C_3} \mathbf{F}\cdot d\mathbf{r}.\end{aligned}$$

Thus,

$$\int_{C_3} \mathbf{F}\cdot d\mathbf{r} = \int_{C_2} \mathbf{F}\cdot d\mathbf{r} = 1.$$

(b) No, if it were then the integral on any closed curve in the region $x^2 + y^2 > 1$ would have to be zero, but $\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = 1$.

(c) We can make the following conditional statement. If P and Q satisfy the assumptions of Green's theorem in the region $x^2 + y^2 \leq 1$, then there must exist a point (x_0, y_0) inside S_2 such that $\frac{\partial Q(x_0,y_0)}{\partial x} \neq \frac{\partial P(x_0,y_0)}{\partial y}$. Otherwise we would be able to apply Green's theorem in the region inside S_2 to conclude that $\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = 0$, contradicting the given assumptions.

Question 5. Use the divergence theorem to evaluate

$$\iint_S \mathbf{F} \cdot d\mathbf{S},$$

where $\mathbf{F}(x, y, z) = 3xy^2\,\mathbf{i} + xe^z\,\mathbf{j} + z^3\,\mathbf{k}$, and S is the surface of the solid bounded by the cylinder $y^2 + z^2 = 1$ and the planes $x = -1$ and $x = 2$.

Solution 5. Compute

$$\operatorname{div} \mathbf{F} = 3y^2 + 0 + 3z^2.$$

By the divergence theorem,

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_D (3y^2 + 3z^2)\, dV.$$

Integrating in cylindrical coordinates with $y = r\cos\theta$, $z = r\sin\theta$, and $x = x$,

$$\begin{aligned}\iiint_D (3y^2 + 3z^2)\, dV &= \int_0^{2\pi} \int_0^1 \int_{-1}^2 (3r^2\cos^2\theta + 3r^2\sin^2\theta) r\, dx\, dr\, d\theta \\ &= 3\int_0^{2\pi} d\theta \int_0^1 r^3\, dr \int_{-1}^2 dx = \frac{9\pi}{2}.\end{aligned}$$

Question 6. Carefully review the examples done in class using the fundamental theorem of line integrals, Green's theorem, Stokes' theorem, and the divergence theorem.