MATH 2300-04
FINAL EXAM

VANDERBILT UNIVERSITY

Directions: Make sure you clearly indicate the pages where your solutions are written. Answers without justification will receive little or no credit. Write clearly, legibly, and in a logical fashion. Make precise statements (for instance, write an equal sign if two expressions are equal; say that one expression is a consequence of another when this is the case, etc.).

The notation and conventions in this exam are the same as used in class, unless stated otherwise.

If you need to use a theorem or formula that was stated in class, you do not need to prove it, unless a question explicitly says so. You do need, however, to state the theorems or formulas you invoke.

If you do not understand a question, or think that some problem is ambiguous, missing information, or incorrectly stated, write how you interpret the problem and solve it accordingly.

Question	Points
1 (15 pts)	
2 (15 pts)	
3 (15 pts)	
4 (15 pts)	
5 (20 pts)	
6 (20 pts)	
Cheat sheet (1 pts)	
TOTAL (100 pts)	

Question 1. (15 pts) (a) Use multiple integrals to express the volume of a sphere of radius R as a triple integral.

(a) Use multiple integrals to express the surface area of a sphere of radius R as a double integral.

Solution 1. The sphere is given by $x^2+y^2+z^2=R^2$ in Cartesian coordinates, or by $\rho=R$, $0\le\phi\le\pi$, $0\le\theta\le 2\pi$ in spherical coordinates.

The volume enclosed by the sphere is

$$\begin{aligned}\text{Volume} &= \iiint\limits_V dV \\ &= \iiint\limits_V \rho^2\sin\phi\,d\rho\,d\phi\,d\theta \\ &= \int_0^{2\pi} d\theta \int_0^{\pi}\sin\phi\,d\phi\int_0^R \rho^2\,d\rho.\end{aligned}$$

(b) To compute the surface area, we parametrize the sphere by

$$\mathbf{r}(\phi,\theta) = R\sin\phi\cos\theta\,\mathbf{i} + R\sin\phi\sin\theta\,\mathbf{j} + R\cos\phi\,\mathbf{k},$$

with $0\le\phi\le\pi$ and $0\le\theta\le 2\pi$. Then

$$\mathbf{r}_\phi\times\mathbf{r}_\theta = R^2\sin^2\phi\cos\theta\,\mathbf{i} + R^2\sin^2\phi\sin\theta\,\mathbf{j} + R^2\sin\phi\cos\phi\,\mathbf{k},$$

and

$$|\mathbf{r}_\phi\times\mathbf{r}_\theta| = R^2\sin\phi.$$

The surface are of the sphere is

$$\begin{aligned}\text{Area} &= \iint\limits_S dS \\ &= \iint\limits_D |\mathbf{r}_\phi\times\mathbf{r}_\theta|\,dA \\ &= \int_0^{2\pi}\int_0^{\pi} R^2\sin\phi\,d\phi\,d\theta.\end{aligned}$$

Question 2. (15 pts). Let R be the region in the first quadrant bounded by the curves $y = 3x$, $y = \frac{1}{x}$, $y = 2x$, and $y = \frac{3}{x}$.

(a) Find a change of variables $x = x(u, v)$, $y = y(u, v)$ that transforms the region R into a rectangle in the uv-plane.

(b) Using the transformation you found in part (a), evaluate the integral

$$\iint_R \frac{y}{x}\, dA.$$

Solution 2. (a) The curves and the region R are shown in figure 2.

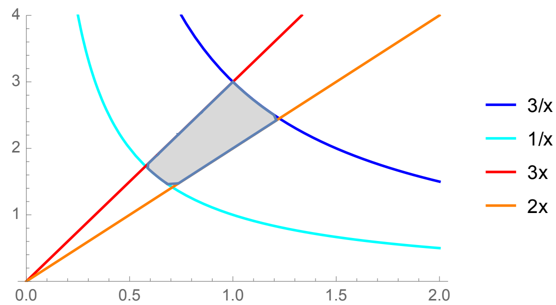

FIGURE 1. The region of question 2.

The curves can be written as $\frac{y}{x} = 3$, $xy = 1$, $\frac{y}{x} = 2$, and $xy = 3$. Set $u = xy$ and $v = \frac{y}{x}$. Then $uv = y^2$ and $\frac{u}{v} = x^2$. In these expressions we chose the positive square root because x and y are positive since they belong to the first quadrant. Thus we find

$$x = \sqrt{\frac{u}{v}} \text{ and } y = \sqrt{uv}.$$

The region in uv-coordinates is given by the rectangle $1 \leq u \leq 3$ and $2 \leq v \leq 3$.

(b) The Jacobian of the transformation is

$$\frac{\partial(x, y)}{\partial(u, v)} = \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \det \begin{bmatrix} \frac{1}{2\sqrt{uv}} & -\frac{\sqrt{u}}{2v^{\frac{3}{2}}} \\ \frac{\sqrt{v}}{2\sqrt{u}} & \frac{\sqrt{u}}{2\sqrt{v}} \end{bmatrix} = \frac{1}{2v}.$$

The change of variable formula now gives

$$\iint_R \frac{y}{x}\, dA = \iint_S \frac{y(u,v)}{x(u,v)} \left| \frac{\partial(x, y)}{\partial(u, v)} \right| \, dA = \int_2^3 \int_1^3 v \frac{1}{2v}\, du\, dv = \frac{1}{2} \int_2^3 \int_1^3 du\, dv = 1.$$

Question 3. (15 pts) Consider the vector field $\mathbf{F}(x, y) = (3 + 2xy)\mathbf{i} + (x^2 - 3y^2)\mathbf{j}$.

(a) Determine whether $\mathbf{F}$ is a conservative vector field. In case yes, find a function f such that $\mathbf{F} = \nabla f$.

(b) Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is the curve given by $\mathbf{r}(t) = \cos(2t)\mathbf{i} + \sin(2t)\mathbf{j}$, $0 \leq t \leq \pi$.

Solution 3. (a)We have $\partial_y P = 2x = \partial_x Q$, so the vector field is conservative. Thus $f_x(x, y) = 3+2xy$, so $f(x, y) = 3x+x^2y+g(y)$. then $f_y(x, y) = x^2+g'(y) = x^2-3y^2$, thus $g(y) = -y^3+K$. We can choose $K = 0$, hence $f(x, y) = 3x + x^2y - y^3$.

(b) The integral is zero by the fundamental theorem of line integrals since the path is a closed loop and $\mathbf{F}$ is conservative.

Question 4. (15 pts) Let S be the part of the cone $z = \sqrt{x^2 + y^2}$ between the planes $z = 1$ and $z = 3$.

(a) Sketch the surface S.

(b) Write S as a parametric surface.

(c) Find an expression for the unit normal vector field to S. Orient it so that the normal points downward.

(d) Evaluate

$$\iint_S \mathbf{F} \cdot d\mathbf{S},$$

where $\mathbf{F}(x, y, z) = -x\,\mathbf{i} - y\,\mathbf{j} + z^3\,\mathbf{k}$.

Solution 4. (a) The surface is part of a cone:

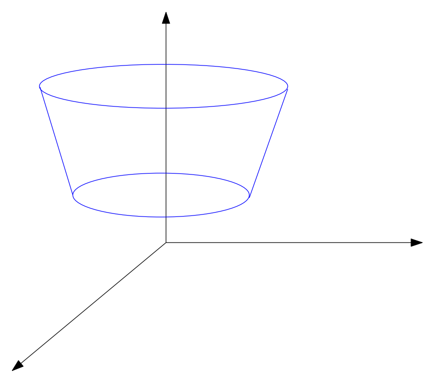

FIGURE 2. The surface of question 4.

(b) The surface S is the graph of the function $z(x, y) = \sqrt{x^2 + y^2}$, $1 \le x^2 + y^2 \le 9$, thus we can write it as a parametric surface by

$$\mathbf{r}(x, y) = x\,\mathbf{i} + y\,\mathbf{j} + \sqrt{x^2 + y^2}\,\mathbf{k}.$$

(c) We have, using that the surface is a graph, that

$$\begin{aligned} \mathbf{r}_x \times \mathbf{r}_y &= -\frac{\partial z}{\partial x}\,\mathbf{i} - \frac{\partial z}{\partial y}\,\mathbf{j} + \mathbf{k} \\ &= -\frac{x}{\sqrt{x^2 + y^2}}\,\mathbf{i} - \frac{y}{\sqrt{x^2 + y^2}}\,\mathbf{j} + \mathbf{k} \end{aligned}$$

is a normal pointing upward, thus

$$\mathbf{n} = \frac{1}{2}\left(\frac{x}{\sqrt{x^2 + y^2}}\,\mathbf{i} + \frac{y}{\sqrt{x^2 + y^2}}\,\mathbf{j} - \mathbf{k}\right).$$

(d) For a graph,

$$\mathbf{F} \cdot \left(\frac{\mathbf{r}_x \times \mathbf{r}_y}{|\mathbf{r}_x \times \mathbf{r}_y|}\right) = \frac{1}{|\mathbf{r}_x \times \mathbf{r}_y|}(P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}) \cdot \left(-\frac{\partial z}{\partial x}\,\mathbf{i} - \frac{\partial z}{\partial y}\,\mathbf{j} + \mathbf{k}\right)$$

when the normal points upward. In our case, following the convention that the normal points downward

$$
\begin{aligned}
\iint_S \mathbf{F} \cdot d\mathbf{S} &= -\iint_D (-P\frac{\partial z}{\partial x} - Q\frac{\partial z}{\partial y} + R)\, dA \\
&= -\iint_D (-(-x)\frac{x}{\sqrt{x^2+y^2}} - (-y)\frac{y}{\sqrt{x^2+y^2}} + (\sqrt{x^2+y^2})^3)\, dA \\
&= -\iint_D (\frac{x^2+y^2}{\sqrt{x^2+y^2}} + (\sqrt{x^2+y^2})^3)\, dA.
\end{aligned}
$$

Using polar coordinates,

$$
\begin{aligned}
\iint_D (\frac{x^2+y^2}{\sqrt{x^2+y^2}} + (\sqrt{x^2+y^2})^3)\, dA &= -\int_0^{2\pi}\int_1^3 (\frac{r^2}{r} + r^3) r\, dr\, d\theta \\
&= -2\pi\, (\frac{1}{3}r^3 + \frac{1}{5}r^5)\Big|_1^3 = -\frac{1712}{15}\pi.
\end{aligned}
$$

Question 5. (20 pts) Let S be a surface that forms a simple plane region on the xy-plane, i.e., S lies on the xy-plane and its boundary is a simple closed curve. Assume that S is oriented with the upward orientation and. Show that in this case Stokes' theorem reduces to Green's theorem. In other words, show that the formula for Stokes' theorem in this case is the same as the formula for Green's theorem.

Solution 5. We have

$$\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \iint_S \operatorname{curl} \mathbf{F} \cdot \mathbf{n}\, dS.$$

Since S is on the xy-plane, the surface is identical to its domain of parametrization, i.e., $S = D$, and $dS = dA$. Moreover, $\mathbf{n} = \mathbf{k}$, so

$$\iint_S \operatorname{curl} \mathbf{F} \cdot \mathbf{n}\, dS = \iint_D \operatorname{curl} \mathbf{F} \cdot \mathbf{k}\, dA.$$

Writing $F = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$, compute

$$\begin{aligned} \operatorname{curl} \mathbf{F} &= \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{bmatrix} \\ &= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathbf{k}, \end{aligned}$$

so that

$$\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \iint_D \operatorname{curl} \mathbf{F} \cdot \mathbf{k}\, dA = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA.$$

On the other hand, Stokes' theorem gives

$$\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r}.$$

Combining the last two equations gives the result.

Question 6. (20 pts) Let

$$\mathbf{F}(x,y,z) = \frac{x}{(x^2+y^2+z^2)^{\frac{3}{2}}}\,\mathbf{i} + \frac{y}{(x^2+y^2+z^2)^{\frac{3}{2}}}\,\mathbf{j} + \frac{z}{(x^2+y^2+z^2)^{\frac{3}{2}}}\,\mathbf{k}.$$

Let S be a smooth closed surface. Show that

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = 4\pi$$

if S encloses the origin, and

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = 0$$

otherwise.

You can use that $\operatorname{div}\mathbf{F} = 0$ for $(x,y,z) \neq (0,0,0)$ (i.e., you do not need to show that the divergence of $\mathbf{F}$ is zero). Notice, also, that if we write $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$, then $\mathbf{F}$ can be written as

$$\mathbf{F} = \frac{\mathbf{r}}{|\mathbf{r}|^3}.$$

Solution 6. On any region not containing the origin, $\operatorname{div}\mathbf{F} = 0$, thus the divergence theorem gives $\iint_S \mathbf{F}\cdot d\mathbf{S} = 0$. If S bounds a region containing the origin, we cannot apply the divergence theorem because the vector field is not defined at the origin. Let S_r be a sphere of radius r centered at the origin and choose r small enough so that S_r is contained in the interior of S. If E is the region between S_r and S, we can apply the divergence theorem there (because $(0,0,0) \notin E$) so that

$$\iiint_E \operatorname{div}\mathbf{F}\,dV = 0 = \iint_{S\cup S_r} \mathbf{F}\cdot d\mathbf{S} = \iint_S \mathbf{F}\cdot d\mathbf{S} + \iint_{S_r} \mathbf{F}\cdot d\mathbf{S}.$$

Denote by $\mathbf{n}$ the unit normal pointing outward on the surface of E. Choosing the normal on S_r to point inward (which corresponds to the normal to pointing outward on the surface of E) we find

$$\iint_{S_r} \mathbf{F}\cdot d\mathbf{S} = -\iint_{S_r} \mathbf{F}\cdot\mathbf{n}\,dS = -\iint_{S_r} \frac{\mathbf{r}}{|\mathbf{r}|^3}\cdot\frac{\mathbf{r}}{|\mathbf{r}|}\,dS = -\frac{1}{r^2}\iint_{S_r} dS = -4\pi.$$

Thus

$$\iint_{S\cup S_r} \mathbf{F}\cdot d\mathbf{S} = 4\pi.$$

Cheat sheet. (1 pts) Do not forget that you must upload your cheat sheet.