VANDERBILT UNIVERSITY, MATH 2300-04, F 20
EXAMPLES OF SECTION 13.2

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Question 1. Using the Fundamental Theorem of Calculus for real-valued functions, prove the Fundamental Theorem of Calculus for vector-valued functions.

Solution 1. Let $\mathbf{r}$ be a vector-valued function defined on some interval containing the points a and b. Assume that there exists a differentiable vector valued function $\mathbf{R}$ such that $\mathbf{r}(t) = \mathbf{R}'(t)$ for all $a \leq t \leq b$. Writing $\mathbf{R} = \langle R_1, R_2, R_3 \rangle$, we have

$$\mathbf{R}' = R_1'\mathbf{i} + R_2'\mathbf{j} + R_3\mathbf{k},$$

so that

$$\begin{aligned} \int_a^b \mathbf{r}(t)\,dt &= \int_a^b \mathbf{R}'(t)\,dt. \\ &= (\int_a^b R_1'(t)\,dt)\mathbf{i} + (\int_a^b R_2'(t)\,dt)\mathbf{j} + (\int_a^b R_3'(t)\,dt)\mathbf{k} \end{aligned}$$

By the Fundamental Theorem of Calculus for real-valued functions,

$$\int_a^b R_1'(t)\,dt = R_1(b) - R_1(a),$$

and similarly for R_2 and R_3. Therefore

$$\begin{aligned} \int_a^b \mathbf{r}(t)\,dt &= (R_1(b) - R_1(a))\mathbf{i} + (R_2(b) - R_2(a))\mathbf{j} + (R_3(b) - R_3(a))\mathbf{k} \\ &= \mathbf{R}(b) - \mathbf{R}(a). \end{aligned}$$