VANDERBILT UNIVERSITY

MATH 2300 – MULTIVARIABLE CALCULUS

Practice Final

Directions. This practice test should be used as a study guide, illustrating the concepts that will be emphasized in the test. This does not mean that the actual test will be restricted to the content of the practice. Try to identify, from the questions below, the concepts and sections that you should master for the test. For each question in the practice test, study the ideas and techniques connected to the problem, even if they are not directly used in your solution.

Take this also as an opportunity to practice how you will write your solutions in the test. For this, write clearly, legibly, and in a logical fashion. Make precise statements (for instance, write an equal sign if two expressions are equal; say that one expression is a consequence of another when this is the case, etc).

Question 1. Find the limits or show that they do not exist.

(a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^4}{x^2+y^8}$.

(b) $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{x^2y^2z^2}{x^2+y^2+z^2}$.

Question 2. Express the given integral as an iterated integral in Cartesian coordinates in six different ways.

(a) $$\iiint_D f(x, y, z)\, dV,$$

where D is the solid bounded by $y = 4 - x^2 - 4z^2$ and $y = 0$.

(b) $$\int_0^1 \int_y^1 \int_0^y f(x, y, z)\, dz\, dx\, dy.$$

Question 3. Use multiple integrals to compute the volume and the surface area of a sphere of radius R.

Question 4. Let R be the region in the xy-plane bounded by the lines $x = 2y$, $x = 2y + 4$, $3x = y + 1$, and $3x = y + 8$.

(a) Find a change of variables that maps a rectangular region S in the uv-plane onto R, where the sides of S are parallel to the $u-$ and $v-$axes.

(b) Use your answer in part (a) to evaluate the integral

$$\iint_R \frac{x - 2y}{3x - y}\, dA.$$

Question 5. Let $\mathbf{F}(x,y) = -\frac{y}{x^2+y^2}\,\mathbf{i} + \frac{x}{x^2+y^2}\,\mathbf{j}$. Show that

$$\int_C \mathbf{F} \cdot d\mathbf{r} = 2\pi,$$

for every simple curve C that encloses the origin and is oriented counterclockwise.

Question 6. (a) State Green's theorem.

(b) Use Green's theorem to show that the area of a planar region D is given by

$$A = \int_{\partial D} x\,dy = -\int_{\partial D} y\,dx = \frac{1}{2}\int_{\partial D} (x\,dy - y\,dx).$$

(c) Suppose that the vertices of a polygon, in counterclockwise order, are (x_1, y_1), (x_2, y_2), $\ldots$, (x_n, y_n). Show that the area of the polygon is

$$A = \frac{1}{2}\Big((x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + \cdots + (x_{n-1}y_n - x_ny_{n-1}) + (x_ny_1 - x_1y_n)\Big).$$

Question 7. Evaluate

$$\iint_S \mathbf{F} \cdot d\mathbf{S},$$

where $\mathbf{F}(x, y, z) = -x\,\mathbf{i} - y\,\mathbf{j} + z^3\,\mathbf{k}$, and S is the part of the cone $z = \sqrt{x^2 + y^2}$ between the planes $z = 1$ and $z = 3$.

Question 8. (a) State Stoke's theorem.

(b) Use Stoke's theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = xy\,\mathbf{i} + yz\,\mathbf{j} + zx\,\mathbf{k}$ and C is the boundary part of the paraboloid $z = 1 - x^2 - y^2$ in the first octant.

Question 9. (a) State the divergence theorem.

(b) Use the divergence theorem to evaluate

$$\iint_S \mathbf{F} \cdot d\mathbf{S},$$

where $\mathbf{F}(x, y, z) = 3xy^2\,\mathbf{i} + xe^z\,\mathbf{j} + z^3\,\mathbf{k}$, and S is the surface of the solid bounded by the cylinder $y^2 + z^2 = 1$ and the planes $x = -1$ and $x = 2$.

Question 10. Evaluate

$$\iint_S \mathbf{F} \cdot d\mathbf{S},$$

where

$$\mathbf{F}(\mathbf{r}) = \frac{\mathbf{r}}{|\mathbf{r}|^3},$$

and S is any closed surface that encloses the origin.