VANDERBILT UNIVERSITY

MATH 2300 – MULTIVARIABLE CALCULUS

Solutions to the Practice Final

Directions. This practice test should be used as a study guide, illustrating the concepts that will be emphasized in the test. This does not mean that the actual test will be restricted to the content of the practice. Try to identify, from the questions below, the concepts and sections that you should master for the test. For each question in the practice test, study the ideas and techniques connected to the problem, even if they are not directly used in your solution.

Take this also as an opportunity to practice how you will write your solutions in the test. For this, write clearly, legibly, and in a logical fashion. Make precise statements (for instance, write an equal sign if two expressions are equal; say that one expression is a consequence of another when this is the case, etc).

Question 1. Find the limits or show that they do not exist.

(a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^4}{x^2+y^8}$.

(b) $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{x^2y^2z^2}{x^2+y^2+z^2}$.

Solution 1. (a) Let $f(x,y) = \frac{xy^4}{x^2+y^8}$. Since $f(x,0) = 0$ for $x \neq 0$, $f(x,y) \to 0$ as $(x,y) \to (0,0)$ along the x-axis. Along the curve $x = y^4$, $f(x,y) = f(y^4, y) = 1/2$ for $y \neq 0$, so $f(x,y) \to 1/2$ along this curve and the limit does not exist.

(b) We have

$$0 \leq \frac{x^2y^2z^2}{x^2+y^2+z^2} \leq x^2y^2 \to 0 \text{ as } (x,y,z) \to (0,0,0),$$

thus the limit is zero by the squeeze theorem.

Question 2. Express the given integral as an iterated integral in Cartesian coordinates in six different ways.

(a) $\displaystyle\iiint_D f(x,y,z)\,dV,$

where D is the solid bounded by $y = 4 - x^2 - 4z^2$ and $y = 0$.

(b) $\displaystyle\int_0^1 \int_y^1 \int_0^y f(x,y,z)\,dz\,dx\,dy.$

Solution 2. (a) The region of integration can be described as

$$
\begin{aligned}
D &= \{-2 \le x \le 2, 0 \le y \le 4 - x^2, -\frac{1}{2}\sqrt{4 - x^2 - y} \le z \le \frac{1}{2}\sqrt{4 - x^2 - y}\} \\
&= \{0 \le y \le 4, -\sqrt{4 - y} \le x \le \sqrt{4 - y}, -\frac{1}{2}\sqrt{4 - x^2 - y} \le z \le \frac{1}{2}\sqrt{4 - x^2 - y}\} \\
&= \{-1 \le z \le 1, 0 \le y \le 4 - 4z^2, -\sqrt{4 - y - 4z^2} \le x \le \sqrt{4 - y - 4z^2}\} \\
&= \{0 \le y \le 4, -\frac{1}{2}\sqrt{4 - y} \le z \le \frac{1}{2}\sqrt{4 - y}, -\sqrt{4 - y - 4z^2} \le x \le \sqrt{4 - y - 4z^2}\} \\
&= \{-2 \le x \le 2, -\frac{1}{2}\sqrt{4 - x^2} \le z \le \frac{1}{2}\sqrt{4 - x^2}, 0 \le y \le 4 - x^2 - 4z^2\} \\
&= \{-1 \le z \le 1, \sqrt{4 - 4z^2} \le x \le \sqrt{4 - 4z^2}, 0 \le y \le 4 - x^2 - 4z^2\}.
\end{aligned}
$$

Then

$$
\begin{aligned}
\iiint_D f(x,y,z)\,dV &= \int_{-2}^{2} \int_0^{4-x^2} \int_{-\frac{1}{2}\sqrt{4-x^2-y}}^{\frac{1}{2}\sqrt{4-x^2-y}} f(x,y,z)\,dz\,dy\,dx \\
&= \int_0^4 \int_{-\sqrt{4-y}}^{\sqrt{4-y}} \int_{-\frac{1}{2}\sqrt{4-x^2-y}}^{\frac{1}{2}\sqrt{4-x^2-y}} f(x,y,z)\,dz\,dx\,dy \\
&= \int_{-1}^{1} \int_0^{4-4z^2} \int_{-\sqrt{4-y-4z^2}}^{\sqrt{4-y-4z^2}} f(x,y,z)\,dx\,dy\,dz \\
&= \int_0^4 \int_{-\frac{1}{2}\sqrt{4-y}}^{\frac{1}{2}\sqrt{4-y}} \int_{-\sqrt{4-y-4z^2}}^{\sqrt{4-y-4z^2}} f(x,y,z)\,dx\,dz\,dy \\
&= \int_{-2}^{2} \int_{-\frac{1}{2}\sqrt{4-x^2}}^{\frac{1}{2}\sqrt{4-x^2}} \int_0^{4-x^2-4z^2} f(x,y,z)\,dy\,dz\,dx \\
&= \int_{-1}^{1} \int_{\sqrt{4-4z^2}}^{\sqrt{4-4z^2}} \int_0^{4-x^2-4z^2} f(x,y,z)\,dy\,dx\,dz.
\end{aligned}
$$

(b) From the limits of integration we see that the region of integration is the tetrahedron with vertices at $(0,0,0)$, $(1,1,1)$, $(1,1,0)$, and $(1,0,0)$, i.e.,

$$D = \{0 \le y \le 1, y \le x \le 1, 0 \le z \le y\}.$$

The other five ways to describe the region are

$$\begin{aligned} D &= \{0 \le x \le 1, 0 \le y \le x, 0 \le z \le y\} \\ &= \{0 \le y \le 1, 0 \le z \le y, y \le x \le 1\} \\ &= \{0 \le z \le 1, z \le y \le 1, y \le x \le 1\} \\ &= \{0 \le x \le 1, 0 \le z \le x, z \le y \le x\} \\ &= \{0 \le z \le 1, z \le x \le 1, z \le y \le x\}. \end{aligned}$$

Thus,

$$\begin{aligned} \int_0^1 \int_y^1 \int_0^y f(x,y,z)\,dz\,dx\,dy &= \int_0^1 \int_0^x \int_0^y f(x,y,z)\,dz\,dy\,dx \\ &= \int_0^1 \int_0^y \int_y^1 f(x,y,z)\,dx\,dz\,dy \\ &= \int_0^1 \int_z^1 \int_y^1 f(x,y,z)\,dx\,dy\,dz \\ &= \int_0^1 \int_0^x \int_z^x f(x,y,z)\,dy\,dz\,dx \\ &= \int_0^1 \int_z^1 \int_z^x f(x,y,z)\,dy\,dx\,dz. \end{aligned}$$

Question 3. Use multiple integrals to compute the volume and the surface area of a sphere of radius R.

Solution 3. The sphere is given by $x^2 + y^2 + z^2 = R^2$ in Cartesian coordinates, or by $\rho = R$, $0 \le \phi \le \pi$, $0 \le \theta \le 2\pi$ in spherical coordinates.

The volume enclosed by the sphere is

$$\begin{aligned}\text{Volume} &= \iiint\limits_V dV \\ &= \iiint\limits_V \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta \\ &= \int_0^{2\pi} d\theta \int_0^{\pi} \sin\phi \, d\phi \int_0^R \rho^2 \, d\rho \\ &= 2\pi \, (-\cos\phi)\big|_0^{\pi} \, \frac{\rho^3}{3}\bigg|_0^R \\ &= \frac{4}{3}\pi R^3.\end{aligned}$$

To compute the surface area, we parametrize the sphere by

$$\mathbf{r}(\phi, \theta) = R\sin\phi\cos\theta \, \mathbf{i} + R\sin\phi\sin\theta \, \mathbf{j} + R\cos\phi \, \mathbf{k},$$

with $0 \le \phi \le \pi$ and $0 \le \theta \le 2\pi$. Then

$$\mathbf{r}_\phi \times \mathbf{r}_\theta = R^2\sin^2\phi\cos\theta \, \mathbf{i} + R^2\sin^2\phi\sin\theta \, \mathbf{j} + R^2\sin\phi\cos\phi \, \mathbf{k},$$

and

$$|\mathbf{r}_\phi \times \mathbf{r}_\theta| = R^2 \sin\phi.$$

The surface are of the sphere is

$$\begin{aligned}\text{Area} &= \iint\limits_S dS \\ &= \iint\limits_D |\mathbf{r}_\phi \times \mathbf{r}_\theta| \, dA \\ &= \int_0^{2\pi} \int_0^{\pi} R^2 \sin\phi \, d\phi \, d\theta \\ &= 2\pi R^2 \, (-\cos\phi)\big|_0^{\pi} \\ &= 4\pi R^2.\end{aligned}$$

Question 4. Let R be the region in the xy-plane bounded by the lines $x = 2y$, $x = 2y + 4$, $3x = y + 1$, and $3x = y + 8$.

(a) Find a change of variables that maps a rectangular region S in the uv-plane onto R, where the sides of S are parallel to the $u-$ and $v-$axes.

(b) Use your answer in part (a) to evaluate the integral

$$\iint\limits_R \frac{x - 2y}{3x - y}\, dA.$$

Solution 4. (a) Let $u = x - 2y$ and $v = 3x - y$. Then $x = \frac{1}{5}(2v - u)$ and $y = \frac{1}{5}(v - 3u)$, and R is the image of the rectangle $0 \le u \le 4$, $1 \le v \le 8$.

(b) We compute

$$\frac{\partial(x, y)}{\partial(u, v)} = \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \det \begin{bmatrix} -\frac{1}{5} & \frac{2}{5} \\ -\frac{3}{5} & \frac{1}{5} \end{bmatrix} = \frac{1}{5},$$

so that

$$\begin{aligned} \iint\limits_R \frac{x - 2y}{3x - y}\, dA &= \int_S \frac{x(u, v) - 2y(u, v)}{3x(u, v) - y(u, v)} \left| \frac{\partial(x, y)}{\partial(u, v)} \right| dA \\ &= \frac{1}{5} \int_0^4 \int_1^8 \frac{u}{v}\, dv\, du \\ &= \frac{1}{5} \int_0^4 u\, du \int_1^8 \frac{1}{v}\, dv \\ &= \frac{8}{5} \ln 8. \end{aligned}$$

Question 5. Let $\mathbf{F}(x, y) = -\frac{y}{x^2+y^2}\,\mathbf{i} + \frac{x}{x^2+y^2}\,\mathbf{j}$. Show that

$$\int_C \mathbf{F}\cdot d\mathbf{r} = 2\pi,$$

for every simple curve C that encloses the origin and is oriented counterclockwise.

Solution 5. Write $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$. For $(x, y) \neq (0, 0)$,

$$\frac{\partial Q(x,y)}{\partial x} = \frac{y^2 - x^2}{(x^2+y^2)^2} = \frac{\partial P(x,y)}{\partial y},$$

and the hypotheses of Green's theorem hold in any region that does not contain the origin. Let C_r be a circle of radius r oriented counterclockwise, centered at the origin and contained in inside the region bounded by C. Let D be the region between C_r and C, so that $\partial D = (-C_r) \cup C$. By Green's theorem

$$\int_{\partial D} (P\,dx + Q\,dy) = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = 0.$$

But

$$\begin{aligned}\int_{\partial D} (P\,dx + Q\,dy) &= \int_C (P\,dx + Q\,dy) + \int_{-C_r} (P\,dx + Q\,dy)\\ &= \int_C \mathbf{F}\cdot d\mathbf{r} - \int_{C_r} \mathbf{F}\cdot d\mathbf{r},\end{aligned}$$

so that

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_{C_r} \mathbf{F}\cdot d\mathbf{r}.$$

Parametrizing C_r by $\mathbf{r}(t) = r\cos t\,\mathbf{i} + r\sin t\,\mathbf{j}$, $0 \le t \le 2\pi$, we find

$$\begin{aligned}\int_{C_r} \mathbf{F}\cdot d\mathbf{r} &= \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t))\cdot \mathbf{r}'(t)\,dt\\ &= \int_0^{2\pi} \frac{(-r\sin t)(-r\sin t) + r\cos t\, r\cos t}{r^2}\,dt = 2\pi.\end{aligned}$$

Question 6. (a) State Green's theorem.

(b) Use Green's theorem to show that the area of a planar region D is given by

$$A = \int_{\partial D} x\,dy = -\int_{\partial D} y\,dx = \frac{1}{2}\int_{\partial D}(x\,dy - y\,dx).$$

(c) Suppose that the vertices of a polygon, in counterclockwise order, are (x_1, y_1), (x_2, y_2), $\ldots$, (x_n, y_n). Show that the area of the polygon is

$$A = \frac{1}{2}\Big((x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + \cdots + (x_{n-1}y_n - x_ny_{n-1}) + (x_ny_1 - x_1y_n)\Big).$$

Solution 6. (a) Page 1136 of the textbook.

(b) We know that

$$A = \iint_D dA.$$

To apply Green's theorem, we seek functions P and Q such that

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1.$$

We can choose $P(x, y) = 0$ and $Q(x, y) = x$, in which case

$$A = \iint_D dA = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = \int_{\partial D}(P\,dx + Q\,dy) = \int_{\partial D} x\,dy.$$

Alternatively, we can also choose $P(x, y) = -y$ and $Q(x, y) = 0$, in which case

$$A = \iint_D dA = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = \int_{\partial D}(P\,dx + Q\,dy) = -\int_{\partial D} y\,dx.$$

Yet another possibility is $P(x, y) = -\frac{1}{2}y$ and $Q(x, y) = \frac{1}{2}x$, so that

$$A = \iint_D dA = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = \int_{\partial D}(P\,dx + Q\,dy) = \frac{1}{2}\int_{\partial D}(x\,dy - y\,dx).$$

(c) Consider two consecutive vertices (x_i, y_i) and (x_{i+1}, y_{i+1}). Let C_i be the line segment joining (x_i, y_i) to (x_{i+1}, y_{i+1}), $i = 1, \ldots, n$, with the convention that $(x_{n+1}, y_{n+1}) = (x_1, y_1)$. Let us compute

$$\int_{C_i}(x\,dy - y\,dx).$$

To do so, we can parametrize C_i as

$$\mathbf{r}_i = \big((1-t)x_i + tx_{i+1}\big)\mathbf{i} + \big((1-t)y_i + ty_{i+1}\big)\mathbf{j},\ 0 \le t \le 1.$$

Then, in parametric form,

$$x = (1-t)x_i + tx_{i+1},$$

so that

$$dx = (x_{i+1} - x_i)\,dt,$$

and

$$y = (1-t)y_i + ty_{i+1},$$

so that

$$dy = (y_{i+1} - y_i)\,dt.$$

Therefore,

$$\begin{aligned}\int_{C_i} (x\,dy - y\,dx) &= \int_0^1 \Big(((1-t)x_i + tx_{i+1})(y_{i+1} - y_i) + ((1-t)y_i + ty_{i+1})(x_{i+1} - x_i)\Big)\,dt \\ &= \int_0^1 (x_i y_{i+1} - x_{i+2} y_i)\,dt \\ &= x_i y_{i+1} - x_{i+1} y_i.\end{aligned} \tag{1}$$

Denoting by D the region enclosed by the polygon, from part (b) we have

$$A = \frac{1}{2}\int_{\partial D} (x\,dy - y\,dx) = \frac{1}{2}\sum_{i=1}^{n} \int_{C_i} (x\,dy - y\,dx).$$

Using (1) yields the desired result.

Question 7. Evaluate

$$\iint_S \mathbf{F} \cdot d\mathbf{S},$$

where $\mathbf{F}(x, y, z) = -x\,\mathbf{i} - y\,\mathbf{j} + z^3\,\mathbf{k}$, and S is the part of the cone $z = \sqrt{x^2 + y^2}$ between the planes $z = 1$ and $z = 3$.

Solution 7. The surface S is the graph of the function $z(x, y) = \sqrt{x^2 + y^2}$, $1 \le x^2 + y^2 \le 9$, thus we can write it as a parametrize surface by

$$\mathbf{r}(x, y) = x\,\mathbf{i} + y\,\mathbf{j} + \sqrt{x^2 + y^2}\,\mathbf{k}.$$

Recall that (page 1170 of the textbook)

$$\mathbf{F} \cdot (\mathbf{r}_x \times \mathbf{r}_y) = (P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}) \cdot (-\frac{\partial z}{\partial x}\,\mathbf{i} - \frac{\partial z}{\partial y}\,\mathbf{j} + \mathbf{k})$$

when the normal points upward. In our case, following the convention that the normal points outward, it points downward from the graph, thus,

$$\begin{aligned}
\iint_S \mathbf{F} \cdot d\mathbf{S} &= -\iint_D (-P\frac{\partial z}{\partial x} - Q\frac{\partial z}{\partial y} + R)\,dA \\
&= -\iint_D (-(-x)\frac{x}{\sqrt{x^2+y^2}} - (-y)\frac{y}{\sqrt{x^2+y^2}} + (\sqrt{x^2+y^2})^3)\,dA \\
&= -\iint_D (\frac{x^2+y^2}{\sqrt{x^2+y^2}} + (\sqrt{x^2+y^2})^3)\,dA.
\end{aligned}$$

Using cylindrical coordinates,

$$\begin{aligned}
\iint_D (\frac{x^2+y^2}{\sqrt{x^2+y^2}} + (\sqrt{x^2+y^2})^3)\,dA &= -\int_0^{2\pi}\int_1^3 (\frac{r^2}{r} + r^3) r\,dr\,d\theta \\
&= -2\pi\,(\frac{1}{3}r^3 + \frac{1}{5}r^5)\Big|_1^3 = -\frac{1712}{15}\pi.
\end{aligned}$$

Question 8. (a) State Stokes' theorem.

(b) Use Stoke's theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = xy\,\mathbf{i} + yz\,\mathbf{j} + zx\,\mathbf{k}$ and C is the boundary part of the paraboloid $z = 1 - x^2 - y^2$ in the first octant.

Solution 8. (a) Page 1174 of the textbook.

(b) By Stokes' theorem

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S},$$

where S is any surface satisfying the assumption of Stokes' theorem that has C as boundary. We orient C counterclockwise as seen from above, and take S to be the paraboloid $z = 1 - x^2 - y^2$ oriented upward. Then S is given as a graph over $0 \leq x^2 + y^2 \leq 1$, $x \geq 0$, $y \geq 0$, and thus

$$\operatorname{curl} \mathbf{F} \cdot (\mathbf{r}_x \times \mathbf{r}_y) = (P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}) \cdot (-\frac{\partial z}{\partial x}\,\mathbf{i} - \frac{\partial z}{\partial y}\,\mathbf{j} + \mathbf{k}),$$

where $\operatorname{curl} \mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$. Computing

$$\operatorname{curl} \mathbf{F} = -y\,\mathbf{i} - z\,\mathbf{j} - x\,\mathbf{k},$$

hence

$$\begin{aligned}\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} &= \iint_D (-(-y)(-2x) - (-z)(-2y) + (-x))\,dA \\ &= \iint_D (-2xy - 2y(1 - x^2 - y^2) - x)\,dA.\end{aligned}$$

Integrating in polar coordinates

$$\begin{aligned}\iint_D (-2xy - 2y(1 - x^2 - y^2) - x)\,dA &= \int_0^{\frac{\pi}{2}} \int_0^1 (-2r\cos\theta r\sin\theta - 2r\sin\theta(1 - r^2) - r\cos\theta) r\,dr\,d\theta \\ &= \int_0^{\frac{\pi}{2}} \int_0^1 (-2r^3 \sin\theta\cos\theta - 2(r^3 - r^4)\sin\theta - r^2\cos\theta)\,dr\,d\theta \\ &= \int_0^{\frac{\pi}{2}} (-\frac{r^4}{2}\sin\theta\cos\theta - 2(\frac{r^3}{3} - \frac{r^5}{5})\sin\theta - \frac{r^3}{3}\cos\theta)\Big|_{r=0}^{r=1} d\theta \\ &= \int_0^{\frac{\pi}{2}} (-\frac{1}{2}\sin\theta\cos\theta - \frac{4}{15}\sin\theta - \frac{1}{3}\cos\theta)\,d\theta \\ &= (-\frac{1}{4}\sin^2\theta + \frac{4}{15}\cos\theta - \frac{1}{3}\sin\theta)\Big|_0^{\frac{\pi}{2}} \\ &= -\frac{1}{4} - \frac{4}{15} - \frac{1}{4} = -\frac{17}{20}.\end{aligned}$$

Question 9. (a) State the divergence theorem.

(b) Use the divergence theorem to evaluate

$$\iint_S \mathbf{F} \cdot d\mathbf{S},$$

where $\mathbf{F}(x, y, z) = 3xy^2\,\mathbf{i} + xe^z\,\mathbf{j} + z^3\,\mathbf{k}$, and S is the surface of the solid bounded by the cylinder $y^2 + z^2 = 1$ and the planes $x = -1$ and $x = 2$.

Solution 9. (a) Page 1181 of the textbook.

(b) Compute

$$\operatorname{div} \mathbf{F} = 3y^2 + 0 + 3z^2.$$

By the divergence theorem,

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_D (3y^2 + 3z^2)\, dV.$$

Integrating in cylindrical coordinates with $y = r\cos\theta$, $z = r\sin\theta$, and $x = x$,

$$\begin{aligned} \iiint_D (3y^2 + 3z^2)\, dV &= \int_0^{2\pi} \int_0^1 \int_{-1}^2 (3r^2\cos^2\theta + 3r^2\sin^2\theta) r\, dx\, dr\, d\theta \\ &= 3\int_0^{2\pi} d\theta \int_0^1 r^3\, dr \int_{-1}^2 dx = \frac{9\pi}{2}. \end{aligned}$$

Question 10. Evaluate

$$\iint_S \mathbf{F} \cdot d\mathbf{S},$$

where

$$\mathbf{F}(\mathbf{r}) = \frac{\mathbf{r}}{|\mathbf{r}|^3},$$

and S is any closed surface that encloses the origin.

Solution 10. A direct computation gives

$$\operatorname{div} \mathbf{F} = 0.$$

Let S_r be a sphere of radius r centered at the origin and contained inside the region bounded by S. Then, by the divergence theorem

$$\iiint_D \operatorname{div} \mathbf{F}\, dV = 0 = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS - \iint_{S_r} \mathbf{F} \cdot \mathbf{n}_r\, dS,$$

where $\mathbf{n}$ and $\mathbf{n}_r$ are the unit outer normal to S and S_r, respectively, both pointing outward. Since $\mathbf{n}_r = \frac{\mathbf{r}}{|\mathbf{r}|}$, we find

$$\mathbf{F} \cdot \mathbf{n}_r = \frac{\mathbf{r}}{|\mathbf{r}|^3} \cdot \frac{\mathbf{r}}{|\mathbf{r}|} = \frac{1}{r^2}.$$

Thus,

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_{S_r} \mathbf{F} \cdot \mathbf{n}_r\, dS = \iint_{S_r} \frac{1}{r^2}\, dS = 4\pi.$$