VANDERBILT UNIVERSITY

MATH 2300 – MULTIVARIABLE CALCULUS

Examples of section 16.3

Question 1. Suppose that $\mathbf{F}$ is a vector field of the form

$$\mathbf{F}(\mathbf{r}) = K\frac{\mathbf{r}}{|\mathbf{r}|^3}, \tag{1}$$

where K is a constant and $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Find the work done by $\mathbf{F}$ in moving an object from a point p to a point q along a curve joining p to q in terms of the distances d_1 and d_2 of these points to the origin.

Remark. One important example of a vector field of the form (1) is the gravitational force between two objects of masses m and M, with one of the objects located at the origin, in which case $K = -mMG$, where G is Newton's gravitational constant. Another important example of a vector field of the form (1) is the electric force between two particles of charges q and Q, with one of them located at the origin, in which case $K = \varepsilon qQ$, where ε is Coulomb's constant.

Solution 1. Let us first show that $\mathbf{F}$ is a conservative vector field. Consider the function

$$f(\mathbf{r}) = -\frac{K}{|\mathbf{r}|}, \text{ or, written differently, } f(x,y,z) = -\frac{K}{\sqrt{x^2+y^2+z^2}}.$$

Using the chain rule, we can compute the partial derivatives of f, finding

$$\frac{\partial f(x,y,z)}{\partial x} = \frac{x}{(x^2+y^2+z^2)^{\frac{3}{2}}}, \quad \frac{\partial f(x,y,z)}{\partial y} = \frac{y}{(x^2+y^2+z^2)^{\frac{3}{2}}},$$

and

$$\frac{\partial f(x,y,z)}{\partial z} = \frac{z}{(x^2+y^2+z^2)^{\frac{3}{2}}}.$$

Noting that $\mathbf{F}$ can be written as

$$\mathbf{F}(\mathbf{r}) = \frac{Kx}{(x^2+y^2+z^2)^{\frac{3}{2}}}\mathbf{i} + \frac{Ky}{(x^2+y^2+z^2)^{\frac{3}{2}}}\mathbf{j} + \frac{Kz}{(x^2+y^2+z^2)^{\frac{3}{2}}}\mathbf{k},$$

we conclude that $\mathbf{F} = \nabla f$. Therefore, the integral

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

is independent of paths, and the work done by $\mathbf{F}$ to move an object from p to q does not depend on the curve we chose to join these points. Furthermore, by the fundamental theorem of line integrals,

$$\begin{aligned} W = \int_C \mathbf{F} \cdot d\mathbf{r} &= f(x_2,y_2,z_2) - f(x_1,y_1,z_1) \\ &= -\frac{K}{\sqrt{x_2^2+y_2^2+z_2^2}} + \frac{K}{\sqrt{x_1^2+y_1^2+z_1^2}} \\ &= \frac{K}{d_1} - \frac{K}{d_2}, \text{ where } p = (x_1,y_1,z_1) \text{ and } q = (x_2,y_2,z_2). \end{aligned}$$