

VANDERBILT UNIVERSITY

MATH 2300 – MULTIVARIABLE CALCULUS

Examples of section 15.7

Question 1. Using cylindrical coordinates, find the mass of a ball B given by $x^2 + y^2 + z^2 \leq a^2$ if the density at any point is proportional to its distance from the z -axis.

Solution 1. The density is given by $\rho(x, y, z) = K\sqrt{x^2 + y^2}$, where K is a constant. The mass is given by

$$m = \iiint_B \rho(x, y, z) dV.$$

In cylindrical coordinates,

$$B = \{(r, \theta, z) \mid 0 \leq \theta \leq 2\pi, 0 \leq r \leq a, -\sqrt{a^2 - r^2} \leq z \leq \sqrt{a^2 - r^2}\},$$

so that

$$\begin{aligned} m &= \int_0^{2\pi} \int_0^a \int_{-\sqrt{a^2-r^2}}^{\sqrt{a^2-r^2}} \rho(r \cos \theta, r \sin \theta, z) r \, dz dr d\theta \\ &= 2 \int_0^{2\pi} \int_0^a \int_0^{\sqrt{a^2-r^2}} K r^2 \, dz dr d\theta \\ &= 2 \int_0^{2\pi} \int_0^a K r^2 \sqrt{a^2 - r^2} \, dr d\theta \\ &= 2K \int_0^{2\pi} \left[\frac{1}{8} r(2r^2 - a^2) \sqrt{a^2 - r^2} + \frac{1}{8} a^4 \arcsin\left(\frac{r}{a}\right) \right]_{r=0}^{r=a} d\theta \\ &= K \frac{\pi}{8} a^4 \int_0^{2\pi} d\theta \\ &= \frac{1}{4} a^4 \pi^2 K. \end{aligned}$$