VANDERBILT UNIVERSITY
MATH 198 —METHODS OF ORDINARY DIFFERENTIAL EQUATIONS
SOLUTIONS TO THE PRACTICE MIDTERM.

Question 1. For each equation below, identify the unknown function, classify the equation as linear or non-linear, and state its order.

(a) $y\frac{dy}{dx} + \frac{y}{x} = 0$.

Solution. Unknown: y. Non-linear, first order.

(b) $x'''' + \cos t\, x' = \sin t$.

Solution. Unknown: x. Linear, fourth order.

(c) $y''' = -\cos y\, y'$.

Solution. Unknown: y. Non-linear, third order.

Question 2. Solve the following initial value problems.

(a) $y' = \frac{y-1}{x+3}$, $y(-1) = 0$.

Solution. This equation is separable, so

$$\frac{dy}{y-1} = \frac{dx}{x+3} \Rightarrow \int \frac{dy}{y-1} = \int \frac{dx}{x+3},$$

from what we obtain

$$|y-1| = C|x+3|,$$

or yet

$$y = 1 + C(x+3).$$

Using the initial condition we find $C = -\frac{1}{2}$, thus

$$y = 1 - \frac{1}{2}(x+3).$$

(b) $y' = e^{-x} - 4y$, $y(0) = \frac{4}{3}$.

Solution. This is a linear equation with $p(x) = 4$ and $q(x) = e^{-x}$. Using the formula for first order linear equations, we find

$$e^{\int p(x)\,dx} = e^{4x},$$

so that

$$y = \frac{1}{3}e^{-x} + Ce^{-4x}.$$

The initial condition gives $C = 1$, so

$$y = \frac{1}{3}e^{-x} + e^{-4x}.$$

Question 3. Solve the following differential equations.

(a) $y' = \dfrac{2y}{x} - x^2y^2$.

Solution. This is a Bernoulli equation with $n = 2$. Setting $v = y^{-1}$, so that

$$\frac{dv}{dx} = -y^{-2}\frac{dy}{dx},$$

one obtains

$$\frac{dv}{dx} + 2\frac{v}{x} = x^2.$$

The latter is a linear equation with $p(x) = \frac{2}{x}$ and $q(x) = x^2$. Using the formula for linear first order equations produces

$$v = \frac{x^3}{5} + \frac{C}{x^2}.$$

From this we find

$$y = \frac{5x^2}{x^5 + C}.$$

(b) $x' = \dfrac{x^2 + t\sqrt{t^2 + x^2}}{tx}$.

Solution. Divide the numerator and denominator of the right-hand by t^2 to find

$$x' = \frac{(x/t)^2 + \sqrt{1 + (x/t)^2}}{(x/t)}.$$

This is, therefore, a homogeneous equation, and the substitution $v = \frac{x}{t}$ yields

$$t\frac{dv}{dt} = \frac{\sqrt{1 + v^2}}{v},$$

which is separable and has solution

$$\sqrt{1 + v^2} = \ln|t| + C.$$

Plugging back $v = \frac{x}{t}$,

$$\sqrt{1 + \left(\frac{x}{t}\right)^2} = \ln|t| + C.$$

(c) $y' = \sin(x - y)$.

Solution. Make the substitution $z = x - y$. Then

$$1 - \frac{dz}{dx} = \sin z \Rightarrow \frac{dz}{1 - \sin z} = dx.$$

Since

$$\int \frac{dz}{1-\sin z} = \int \frac{(1+\sin z)dz}{1-\sin^2 z} = \int \frac{(1+\sin z)dz}{\cos^2 z}$$
$$= \int \sec^2 z\, dz + \int \tan z \sec z\, dz = \tan z + \sec z,$$

the solution is given implicitly by

$$\tan(x-y) + \sec(x-y) = x + C.$$

(d) $y' = \dfrac{\cos y \cos x + 2x}{\sin y \sin x + 2y}$.

Solution. Write

$$(\cos y \cos x + 2x)dx - (\sin y \sin x + 2y)dy = 0.$$

We readily verify that this equation is exact, with $M(x,y) = \cos y \cos x + 2x$ and $N(x,y) = -(\sin y \sin x + 2y)$. Then

$$F(x,y) = \int M(x,y)\, dx = \sin x \cos y + x^2 + g(y).$$

From

$$\frac{\partial F}{\partial y} = N,$$

we find

$$g'(y) = -2y,$$

hence $g(y) = -y^2$. The general solution is

$$F(x,y) = \sin x \cos y + x^2 - y^2 = C.$$

Question 4. Find the general solution of the given differential equation.

(a) $y'' + 8y' - 14y = 0$.

Solution. $\lambda^2 + 8\lambda - 14 = 0 \Rightarrow \lambda = -4 \pm \sqrt{30}$. $y = c_1 e^{(-4+\sqrt{30})t} + c_2 e^{(-4-\sqrt{30})t}$.

(b) $y'' + 8y' - 9y = 0$.

Solution. $\lambda^2 + 8\lambda - 9 = 0 \Rightarrow \lambda = -9, \lambda = 1$. $y = c_1 e^{-9t} + c_2 e^{t}$.

(c) $t^2 y'' + 5y = 0$, $t > 0$.

Solution. Cauchy-Euler equation. $\lambda^2 - \lambda + 5 = 0 \Rightarrow \lambda = \frac{1 \pm i\sqrt{19}}{2}$. $y = c_1 t^{\frac{1}{2}} \cos(\frac{\sqrt{19}}{2} \ln t) + c_2 t^{\frac{1}{2}} \sin(\frac{\sqrt{19}}{2} \ln t)$.

Question 5. Give the form of the particular solution for the given differential equations. You do not have to find the values of the constants of the particular solution.

(a) $y'' + 2y' - 3y = \cos x$.

Solution. The characteristic equation is

$$\lambda^2 + 2\lambda - 3 = (\lambda - 1)(\lambda + 3) = 0.$$

Hence $y_1 = e^x$ and $y_2 = e^{-3x}$ are linearly independent solutions of the associated homogeneous equation. Since these do not involve $\cos x$, we have

$$y_p = A\cos x + B\sin x.$$

(b) $y'' + 4y = 8\sin 2t$.

Solution. The characteristic equation is

$$\lambda^2 + 4 = 0.$$

Hence $y_1 = \cos 2t$ and $y_2 = \sin 2t$ are linearly independent solutions of the associated homogeneous equation. Therefore we need to take $s = 1$, so

$$y_p = At\cos t + Bt\sin t.$$

(c) $y'' - 2y' + y = e^t \cos t$.

Solution. The characteristic equation is

$$\lambda^2 - 2\lambda + 1 = (\lambda - 1)^2 = 0.$$

Hence $y_1 = e^t$ and $y_2 = te^t$ are linearly independent solutions of the associated homogeneous equation. Thus,

$$y_p = (A\cos t + B\sin t)e^t.$$

(d) $y'' - y' - 12y = 2t^6 e^{-3t}$.

Solution. The characteristic equation is

$$\lambda^2 - \lambda - 12 = (\lambda + 3)(\lambda - 4) = 0.$$

Hence $y_1 = e^{-3t}$ and $y_2 = e^{4t}$ are linearly independent solutions of the associated homogeneous equation. We need to take $s = 1$, thus

$$y_p = t(a_6t^6 + a_5t^5 + a_4t^4 + a_3t^3 + a_2t^2 + a_2t^1 + a_0)e^{-3t}.$$

Question 6. Verify that the given functions are two linearly independent solutions of the corresponding homogeneous equation. Then, find a particular solution solving the non-homogeneous problem.

(a) $x^2y'' - 2y = 3x^2 - 1$, $x > 0$, $y_1 = x^2$, $y_2 = x^{-1}$.

(b) $(1 - x)y'' + xy' - y = \sin x$, $0 < x < 1$, $y_1 = e^x$, $y_2 = x$.

Solution. The verification is done by plugging in the given functions into the equation, while the particular solution is found with the formula

$$y_p(t) = -y_1(t)\int \frac{y_2(t)f(t)}{W(t)}\,dt + y_2(t)\int \frac{y_1(t)f(t)}{W(t)}\,dt,$$

The important thing to remember here is that in order to use the above formula we need the coefficient of y'' to be equal to one. So, for example, in (b) we need to write

$$y'' + \frac{x}{1-x}y' - \frac{1}{1-x}y = \frac{\sin x}{1-x}.$$

Using the formula we find

$$y_p(t) = -e^x \int \frac{xe^{-x}\sin x}{(1-x)^2}\,dx + x\int \frac{\sin x}{(1-x)^2}\,dx.$$

Question 7. Show that the problem

$$3y' - x^2 + xy^3 = 0,\ y(1) = 6,$$

has a unique solution defined in some neighborhood of $x = 1$.
Solution. Write

$$y' = f(x,y),\ y(1) = 6,$$

where $f(x,y) = \frac{x^2 - xy^3}{3}$. Since $f(x,y)$ and $\partial_y f(x,y) = -xy^2$ are continuous in the neighborhood of $(1,6)$, the result follows from the existence and uniqueness theorem for first order equations.

URL: http://www.disconzi.net/Teaching/MAT198-Spring-14/MAT198-Spring-14.html