VANDERBILT UNIVERSITY
MATH 198 —METHODS OF ORDINARY DIFFERENTIAL EQUATIONS.
PRACTICE MIDTERM II.

The Laplace transform.

The table below indicates the Laplace transform $F(s)$ of the given function $f(t)$.

$f(t)$	$F(s)$
1	$\frac{1}{s}$
t	$\frac{1}{s^2}$
t^n	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
$\cos(kt)$	$\frac{s}{s^2+k^2}$
$\sin(kt)$	$\frac{k}{s^2+k^2}$
$e^{at}\cos(kt)$	$\frac{s-a}{(s-a)^2+k^2}$
$e^{at}\sin(kt)$	$\frac{k}{(s-a)^2+k^2}$

The following are the main properties of the Laplace transform.

Function	Laplace transform
$af(t)+bg(t)$	$aF(s)+bG(s)$
$f'(t)$	$sF(s)-f(0)$
$f''(t)$	$s^2F(s)-sf(0)-f'(0)$
$f^{(n)}(t)$	$s^nF(s)-s^{n-1}f(0)-\cdots-f^{(n-1)}(0)$
$e^{at}f(t)$	$F(s-a)$
$t^nf(t)$	$(-1)^n\frac{d^n}{ds^n}F(s)$
$(f*g)(t)$	$F(s)G(s)$
$u(t-a)$	$\frac{e^{-as}}{s}$

Above, $f*g$ is the convolution of f and g, given by

$$(f*g)(t)=\int_0^t f(t-\tau)g(\tau)\,d\tau,$$

and $u(t-a)$ is given by

$$u(t-a)=\begin{cases}0, & t<a,\\ 1, & t>a.\end{cases}$$

Question 1. Recall that a function f is said to be of exponential order α if there exist positive constants T and M such that

$$|f(t)| \leq M e^{\alpha t}, \text{ for all } t \geq T.$$

Which of the following functions are of exponential order?

(a) t^2.

(b) e^{-t^6}.

(c) $\dfrac{1}{\sqrt{t} + t^2}$.

Solutions. All of them.

Question 2. Determine the Laplace transform of the given functions

(a) $\cos(nt)\cos(mt)$, $n \neq m$.

(b) $t\sin(2t)\sin(5t)$.

Solutions.

(a) Since

$$\cos A \cos B = \frac{\cos(A-B)+\cos(A+B)}{2},$$

we have

$$\begin{aligned}\mathscr{L}\{\cos(nt)\cos(mt)\} &= \mathscr{L}\left\{\frac{\cos(n-m)t+\cos(n+m)t}{2}\right\}\\ &= \frac{1}{2}\left(\frac{s}{s^2+(n-m)^2}+\frac{s}{s^2+(n+m)^2}\right)\\ &= \frac{s(s^2+n^2+m^2)}{(s^2+(n-m)^2)(^2{+}(n+m)^2)}.\end{aligned}$$

(b) Using

$$\sin A \sin B = \frac{\cos(A-B)-\cos(A+B)}{2},$$

and properties of the Laplace transform,

$$\begin{aligned}\mathscr{L}\{t\sin 2t \sin 5t\} &= -\frac{d}{ds}\mathscr{L}\left\{\frac{\cos 3t-\cos 7t}{2}\right\}\\ &= \frac{20(3s^4+58s^2-411)}{(s^2+9)(s^2+49)^2}.\end{aligned}$$

Question 3. Determine $\mathscr{L}^{-1}\{F\}$.

(a) $F(s) = \dfrac{3s+2}{s^2+2s+10}$.

(b) $sF(s) + F(s) = \dfrac{3s^2+5s+3}{s^3}$.

Solutions.

(a) Completing the square,

$$s^2 + 2s + 10 = (s+1)^2 + 9.$$

Write

$$\frac{3s+2}{s^2+2s+10} = A\frac{s+1}{(s+1)^2+9} + B\frac{3}{(s+1)^2+9}.$$

We find $A = 3$ and $B = -\frac{1}{3}$. Thus the answer is

$$3e^{-t}\cos 3t - \frac{1}{3}e^{-t}\sin 3t.$$

(b) We have

$$F(s) = \frac{3s^2+5s+3}{s^3(s+1)} = \frac{A}{s^3} + \frac{B}{s^2} + \frac{C}{s} + \frac{D}{s+1}.$$

We find $A = 3$, $B = 2$, $C = 1$, $D = -1$, and the answer becomes

$$\mathscr{L}^{-1}\left\{\frac{3}{s^3} + \frac{2}{s^2} + \frac{1}{s} - \frac{1}{s+1}\right\} = \frac{3}{2}t^2 + 2t + 1 - e^{-t}.$$

Question 4. Solve the given initial value problem using the method of Laplace transforms.

(a) $\begin{cases} y'' - y' - 2y = 0, \\ y(0) = -2,\ y'(0) = 5. \end{cases}$

(b) $\begin{cases} y'' - 2y' + y = 6t - 2, \\ y(-1) = 3,\ y'(-1) = 7. \end{cases}$

Solutions.

(a) Taking the Laplace transform,

$$s^2Y + 2s - 5 - (sY + 2) - 2Y = 0,$$

thus

$$Y = \frac{7 - 2s}{(s-2)(s+1)} = \frac{1}{s-2} - \frac{3}{s+1}.$$

So

$$y(t) = e^{2t} - 3e^{-t}.$$

(b) Since we need the initial conditions to be at $t = 0$, we shift $w(t) = y(t-1)$, so that

$$w'' - 2w' + w = 6t - 8.$$

Taking the Laplace transform, we find

$$W = \frac{1}{s^2 - 2s + 1}\left(3s + 1 + \frac{6}{s^2} - \frac{8}{s}\right) = \frac{6}{s^2} + \frac{4}{s} - \frac{1}{s-1} + \frac{2}{(s-1)^2}.$$

So

$$w(t) = 6t + 4 - e^t + 2te^t,$$

and

$$y(t) = 6t + 10 + e^{t+1} + 2te^{t+1}.$$

Question 5. Use convolution to obtain a formula for the solution to the given initial value problem, where g is piece-wise continuous on $[0, \infty)$ and of exponential order.

$$\begin{cases} y'' + y = g(t), \\ y(0) = 0,\ y'(0) = 1. \end{cases}$$

Solutions. Taking the Laplace transform,

$$Y(s) = \frac{1}{s^2+1} + \frac{G(s)}{s^2+1}.$$

Thus

$$y(t) = \sin t + \sin t * g(t) = \sin t + \int_0^t \sin(t-\tau) g(\tau)\, d\tau.$$

Question 6. Solve the given or integro-differential equation for $y(t)$.

$$y'(t) - 2\int_0^t e^{t-\tau} y(\tau)\, d\tau = t,$$
$$y(0) = 2.$$

Solutions. Write the equation as

$$y' - 2e^t * y = t.$$

Then, taking the Laplace transform,

$$\left(s - \frac{2}{s-1}\right) Y - 2 = \frac{1}{s^2},$$

which gives

$$Y = \frac{1}{2s^2} - \frac{3}{4s} + \frac{2}{s+1} + \frac{3}{4(s-2)}.$$

Then

$$y(t) = \frac{t}{2} - \frac{3}{4} + 2e^{-t} + \frac{3e^{2t}}{4}.$$

Question 7. Solve the given symbolic initial value problem.

$$\begin{cases} y'' + y = -\delta(t-\pi) + \delta(t-2\pi), \\ y(0) = 0,\ y'(0) = 1. \end{cases}$$

Solutions.

Solve

$$\begin{cases} y'' + y = -\delta(t-\pi), \\ y(0) = 0,\ y'(0) = 0, \end{cases}$$

and

$$\begin{cases} y'' + y = \delta(t-2\pi), \\ y(0) = 0,\ y'(0) = 1 \end{cases}$$

separately. Then add up the two solutions, finding

$$y(t) = (1 + u(t-\pi) + u(t-2\pi))\sin t.$$

Question 8. Determine the radius of convergence of the given power series.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{7n}{n^3+1}(x-4)^n$

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{n^2}{n!}(x+1)^n$

Solutions.

(a) $R = 1$, (b) $R = \infty$.

Question 9. Review the more theoretical aspects of Laplace transform, such as, but not restricted to: when the Laplace transform exists, how to prove its properties, Laplace transform of discontinuous functions and the delta function, etc.