VANDERBILT UNIVERSITY
MATH 198 —METHODS OF ORDINARY DIFFERENTIAL EQUATIONS.
PRACTICE MIDTERM II.

The Laplace transform.

The table below indicates the Laplace transform $F(s)$ of the given function $f(t)$.

$f(t)$	$F(s)$
1	$\frac{1}{s}$
t	$\frac{1}{s^2}$
t^n	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
$\cos(kt)$	$\frac{s}{s^2+k^2}$
$\sin(kt)$	$\frac{k}{s^2+k^2}$
$e^{at}\cos(kt)$	$\frac{s-a}{(s-a)^2+k^2}$
$e^{at}\sin(kt)$	$\frac{k}{(s-a)^2+k^2}$

The following are the main properties of the Laplace transform.

Function	Laplace transform
$af(t)+bg(t)$	$aF(s)+bG(s)$
$f'(t)$	$sF(s)-f(0)$
$f''(t)$	$s^2F(s)-sf(0)-f'(0)$
$f^{(n)}(t)$	$s^nF(s)-s^{n-1}f(0)-\cdots-f^{(n-1)}(0)$
$e^{at}f(t)$	$F(s-a)$
$t^nf(t)$	$(-1)^n\frac{d^n}{ds^n}F(s)$
$(f*g)(t)$	$F(s)G(s)$
$u(t-a)$	$\frac{e^{-as}}{s}$

Above, $f*g$ is the convolution of f and g, given by

$$(f*g)(t)=\int_0^t f(t-\tau)g(\tau)\,d\tau,$$

and $u(t-a)$ is given by

$$u(t-a)=\begin{cases}0, & t<a,\\ 1, & t>a.\end{cases}$$

Question 1. Recall that a function f is said to be of exponential order α if there exist positive constants T and M such that

$$|f(t)| \leq Me^{\alpha t}, \text{ for all } t \geq T.$$

Which of the following functions are of exponential order?

(a) t^2.

(b) e^{-t^6}.

(c) $\dfrac{1}{\sqrt{t}+t^2}$.

Question 2. Determine the Laplace transform of the given functions

(a) $\cos(nt)\cos(mt)$, $n \neq m$.

(b) $t\sin(2t)\sin(5t)$.

Question 3. Determine $\mathscr{L}^{-1}\{F\}$.

(a) $F(s) = \dfrac{3s+2}{s^2+2s+10}$.

(b) $sF(s) + F(s) = \dfrac{3s^2+5s+3}{s^3}$.

Question 4. Solve the given initial value problem using the method of Laplace transforms.

(a) $\begin{cases} y'' - y' - 2y = 0, \\ y(0) = -2,\ y'(0) = 5. \end{cases}$

(b) $\begin{cases} y'' - 2y' + y = 6t - 2, \\ y(-1) = 3,\ y'(-1) = 7. \end{cases}$

Question 5. Use convolution to obtain a formula for the solution to the given initial value problem, where g is piece-wise continuous on $[0, \infty)$ and of exponential order.

$$\begin{cases} y'' + y = g(t), \\ y(0) = 0,\ y'(0) = 1. \end{cases}$$

Question 6. Solve the given or integro-differential equation for $y(t)$.

$$y'(t) - 2\int_0^t e^{t-\tau} y(\tau)\, d\tau = t,$$
$$y(0) = 2.$$

Question 7. Solve the given symbolic initial value problem.

$$\begin{cases} y'' + y = -\delta(t-\pi) + \delta(t-2\pi), \\ y(0) = 0,\ y'(0) = 1. \end{cases}$$

Question 8. Determine the radius of convergence of the given power series.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{7n}{n^3+1}(x-4)^n$

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{n^2}{n!}(x+1)^n$

Question 9. Review the more theoretical aspects of Laplace transform, such as, but not restricted to: when the Laplace transform exists, how to prove its properties, Laplace transform of discontinuous functions and the delta function, etc.