VANDERBILT UNIVERSITY
MATH 198 —METHODS OF ORDINARY DIFFERENTIAL EQUATIONS.
PRACTICE FINAL SOLUTIONS.

The Laplace transform.

The table below indicates the Laplace transform $F(s)$ of the given function $f(t)$.

$f(t)$	$F(s)$
1	$\frac{1}{s}$
t	$\frac{1}{s^2}$
t^n	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
$\cos(kt)$	$\frac{s}{s^2+k^2}$
$\sin(kt)$	$\frac{k}{s^2+k^2}$
$e^{at}\cos(kt)$	$\frac{s-a}{(s-a)^2+k^2}$
$e^{at}\sin(kt)$	$\frac{k}{(s-a)^2+k^2}$

The following are the main properties of the Laplace transform.

Function	Laplace transform
$af(t)+bg(t)$	$aF(s)+bG(s)$
$f'(t)$	$sF(s)-f(0)$
$f''(t)$	$s^2F(s)-sf(0)-f'(0)$
$f^{(n)}(t)$	$s^nF(s)-s^{n-1}f(0)-\cdots-f^{(n-1)}(0)$
$e^{at}f(t)$	$F(s-a)$
$t^nf(t)$	$(-1)^n\frac{d^n}{ds^n}F(s)$
$(f*g)(t)$	$F(s)G(s)$
$u(t-a)$	$\frac{e^{-as}}{s}$

Above, $f*g$ is the convolution of f and g, given by

$$(f*g)(t)=\int_0^t f(t-\tau)g(\tau)\,d\tau,$$

and $u(t-a)$ is given by

$$u(t-a)=\begin{cases}0, & t<a,\\ 1, & t>a.\end{cases}$$

Question 1. Find the general solution of the the differential equations below.

(a) $y' + \dfrac{2xy - 3x^2}{x^2 - 2y^{-3}} = 0.$

Exact equation. Solution:

$$x^2 y - x^3 + y^{-2} = C.$$

(b) $\dfrac{dy}{dx} = 2 - \sqrt{2x - y + 3}.$

Set $z = 2x - y$ so $z' = \sqrt{z + 3}$. Solution:

$$y = 2x + 3 + \frac{1}{4}(x + C)^2.$$

(c) $y' - 4y = 32x^2.$

Linear equation with $p(x) = -4$, $q(x) = 32x^2$. Using the formula for linear equations

$$y = -(1 + 4x + 8x^2) + Ce^{4x}.$$

(d) $y' = \dfrac{x}{y} + \dfrac{y}{x}.$

Homogeneous equation. Set $v = \frac{y}{x}$ to get $v' = \frac{1}{x}$. Solution:

$$y^2 = x^2 \ln x^2 + Cx^2.$$

(e) $y'' - 5y' + 6y = 0.$

Second order linear with constant coefficients. Characteristic roots $\lambda_1 = 2$ and $\lambda_2 = 3$. Solution:

$$y = c_1 e^{2x} + c_2 e^{3x}.$$

(f) $y' + \frac{y}{x} = -\frac{4x}{y^2}$.

Bernoulli with $n = -2$. Set $v = y^{1-n} = y^3$ to get $v' + \frac{3}{x}v = -12x$, which is linear and can be solved with the formula for linear equations. Then $y = v^{\frac{1}{3}}$ is given by

$$y = \left(-\frac{4x^2}{5} + \frac{C}{x^3}\right)^{\frac{1}{3}}.$$

Question 2. Solve the following initial value problems.

(a) $y'' + 9y = 10e^{2t}$, $y(0) = -1$, $y'(0) = 5$.

Taking the Laplace transform and using partial fractions,

$$Y = \frac{10}{(s^2+9)(s-2)} - \frac{s}{s^2+9} + \frac{5}{x^2+9} = \frac{45-23s}{13(s^2+9)} + \frac{10}{13(s-2)}.$$

Then

$$y(t) = \frac{10}{13}e^{2t} - \frac{23}{13}\cos(3t) + \frac{15}{13}\sin(3t).$$

(b) $y'' + 3y' + 4y = u(t-1)$, $y(0) = 0$, $y'(0) = 1$.

Taking the Laplace transform and using partial fractions (use $s^2 + 3s + 4 = (s + 3/2)^2 + (\sqrt{7}/2)^2$),

$$Y = \frac{1}{(s+3/2)^2 + (\sqrt{7}/2)^2} + e^{-s}\left(\frac{1}{4s} - \frac{s+3/2}{4((s+3/2)^2 + (\sqrt{7}/2)^2)} - \frac{3}{4\sqrt{7}}\frac{\sqrt{7}/2}{(s+3/2)^2 + (\sqrt{7}/2)^2}\right).$$

Then

$$y(t) = \left(\frac{1}{4} - \frac{1}{4}e^{-3(t-1)/2}\cos(\sqrt{7}(t-1)/2) + \frac{3}{4\sqrt{7}}e^{-3(t-1)/2}\sin(\sqrt{7}(t-1)/2)\right)u(t-1)$$
$$+\frac{2}{\sqrt{7}}e^{\frac{-3t}{2}}\sin(\sqrt{7}t/2).$$

(c) $y(t) + \int_0^t y(\tau)(t-\tau)\,d\tau = e^{-3t}$.

The equation is $y + t * y = e^{-3t}$. Taking the Laplace transform and using partial fractions,

$$Y = \frac{s^2}{(s+3)(s^2+1)} = \frac{9}{10(s+3)} + \frac{1}{10(s^2+1)} - \frac{3}{10(s^2+1)}.$$

Then

$$y(t) = \frac{9}{10}e^{-3t} + \frac{1}{10}\cos t - \frac{3}{10}\sin t.$$

Question 3. Using power series, find the general solution to the differential equations below (your solution should include the general form of the coefficients a_n).

(a) $(1 - x^2)y'' + xy' + 3y = 0$.

Set

$$y = \sum_{n=0}^{\infty} a_n x^n.$$

Plugging in,

$$2a_2 + 3a_0 + (6a_3 + 4a_1)x + \sum_{n=2}^{\infty} \left((n+2)(n+1)a_{n+2} - (n-3)(n+1)a_n\right) x^n = 0.$$

So

$$a_{n+2} = \frac{n-3}{n+2} a_n,$$

which gives $a_{2n+1} = 0$ for $n > 1$ and

$$a_{2n} = \frac{(-3)(-1)(1)\cdots(2n-5)}{2^n n!}.$$

Solution:

$$y = a_0 \left(1 + \sum_{n=1}^{\infty} \frac{(-3)(-1)(1)\cdots(2n-5)}{2^n n!} x^{2n}\right) + a_1 \left(x - \frac{2x^3}{3}\right).$$

(b) $(x^2 - 2)y'' + 3y = 0$.

Solution:

$$y = a_0 \left(1 + \sum_{n=1}^{\infty} \frac{2 \cdot 5 \cdot 15 \cdots (4n^2 - 10n + 9)}{2^n (2n)!} x^{2n}\right)$$
$$+ a_1 \left(x + \sum_{n=1}^{\infty} \frac{3 \cdot 9 \cdot 23 \cdots (4n^2 - 6n + 5)}{2^n (2n+1)!} x^{2n+1}\right).$$

Question 4. Find a power series solution to the differential equations below about the given point (your solution should include the general form of the coefficients a_n).

(a) $4x^2 y'' + 2x^2 y' - (x+3)y = 0$, $x > 0$, about $x = 0$.

Use the Frobenius method. Write

$$y'' + \frac{2}{4} y' - \frac{(x+3)}{4x^2} y = 0,$$

to find

$$p_0 = \lim_{x \to 0} x p(x) = 0,\ p_0 = \lim_{x \to 0} x^2 q(x) = -\frac{3}{4}.$$

Then $\lambda(\lambda - 1) + p_0 \lambda + q_0 = 0$ gives $\lambda = \frac{3}{2}$ and $\lambda = -\frac{1}{2}$. We have to use the largest root, $\lambda = \frac{3}{2}$.

Set

$$y = \sum_{n=0}^{\infty} a_n x^{n+\lambda}.$$

Plugging in,

$$4\left((\lambda - 1)\lambda - \frac{3}{4}\right) a_0 + \sum_{n=1}^{\infty} \left(4(n+\lambda-1)(n+\lambda)a_n + 2(n+\lambda-1)a_{n-1} - a_{n-1} - 3a_n\right) x^{n+\lambda} = 0.$$

This gives

$$a_n = \frac{3 - 2n - 2\lambda}{4(n + \lambda - 1)(n + \lambda) - 3} a_{n-1}.$$

Using $\lambda = \frac{3}{2}$ we find

$$a_n = \frac{(-1)^n}{2^{n-1}(n + 2)!} a_0,$$

and

$$y = a_0 x^{\frac{3}{2}} + a_0 \sum_{n=1}^{\infty} \frac{(-1)^n}{2^{n-1}(n + 2)!} x^{n+\frac{3}{2}}.$$

(b) $xy'' + (x - 1)y' - 2y = 0$, $x > 0$, about $x = 0$.

The indicial equation gives $\lambda = 0$ and $\lambda = 2$. Use the largest root.

$$y = \sum_{n=0}^{\infty} a_n x^{n+\lambda}.$$

Plugging in gives

$$(\lambda(\lambda - 1) - \lambda) a_0 x^{\lambda-1} + \sum_{n=1}^{\infty} \left((n + \lambda + 1)(n + \lambda - 1)a_{n+1} + (n + \lambda - 2)a_n\right) x^{n+\lambda} = 0.$$

Using $\lambda = 2$, we get $a_{n+1} = \frac{na_n}{(n+3)(n+1)}$, which gives $a_n = 0$ for all $n \geq 1$. Thus

$$y = a_0 x^2.$$

Question 5. Find at least the first three nonzero terms in the series expansion about $x = 0$ for a general solution of

$$xy'' - y' - xy = 0,\ x > 0.$$

Using the Frobenius method, we find $\lambda_1 = 0$ and $\lambda_2 = 2$. The solution corresponding to the largest root, $\lambda_2 = 2$, is

$$y_1 = a_0 \sum_{n=0}^{\infty} \frac{1}{2^{2n}(n+1)!n!} x^{2n+2} = a_0 \left(x^2 + \frac{1}{8}x^4 + \frac{1}{192}x^6 + \cdots \right).$$

Since the difference $\lambda_2 - \lambda_1 = 0$ is an integer, we seek a second solution of the form

$$y_2 = Cy_1 \ln x + \sum_{n=0} b_n x^{n+\lambda_1}.$$

Plugging in, using $\lambda_1 = 0$, and noticing that the terms in $\ln x$ will cancel out because y_1 is a solution, we find

$$2C\left(y_1' - \frac{y_1}{x}\right) + \sum_{n=2}^{\infty} n(n-1)b_n x^{n-1} - \sum_{n=1}^{\infty} nb_n x^{n-1} - \sum_{n=0}^{\infty} b_n x^{n+1} = 0.$$

Set $a_0 = 1$ and plug y_1 into the above expression to find

$$\begin{gathered} 2C\left(2x + \frac{1}{2}x^3 + \frac{6}{192}x^5 + \cdots - x - \frac{1}{8}x^3 - \frac{1}{192}x^5 - \cdots\right) \\ +2b_2x + 3\cdot 2b_3x^2 + 4\cdot 3b_4x^3 + 5\cdot 4b_5x^4 + 6\cdot 5b_6x^5 + \cdots \\ -b_1 - 2b_2x - 3b_3x^2 - 4b_4x^3 - 5b_5x^4 - 6b_6x^5 - \cdots \\ -b_0x - b_1x^2 - b_2x^3 - b_3x^4 - b_4x^5 - \cdots = 0. \end{gathered}$$

With $C = 1$ we obtain

$$\begin{gathered} \text{terms in } x^0 : b_1 = 0. \\ \text{terms in } x^1 : 2 - b_0 + 2b_2 - 2b_2 = 2 - b_0 = 0 \Rightarrow b_0 = 2. \\ \text{terms in } x^2 : 3\cdot 2b_3 - 3b_3 - b_1 = 0 \Rightarrow b_3 = 0. \\ \text{terms in } x^3 : 2\cdot\frac{1}{2} - 2\frac{1}{8} + 4\cdot 3b_4 - 4b_4 - b_2 = 0 \Rightarrow b_4 = -\frac{3}{32} + b_2. \\ \text{etc.} \end{gathered}$$

Since b_2 is undetermined and we had set $C = 1$, we may set $b_2 = 0$. We finally get

$$y_2 = y_1 \ln x + 2 - \frac{3}{32}x^4 - \frac{7}{1152}x^6 + \cdots$$

Question 6. Find the general solution of the system

$$x' = Ax$$

for the given matrices A.

(a) $A = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix}$.

Solution:

$$x = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 e^{-5t} \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

(b) $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$.

Solution:

$$x = c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \cos t \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} - c_2 \sin t \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + c_3 \sin t \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + c_3 \cos t \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Question 7. Let $F(s) = \mathscr{L}\{f\}(s)$ exist for $s > \alpha$, $\alpha \geq 0$. Show that if $a > 0$, then

$$\mathscr{L}^{-1}\{e^{-as}\} = f(t-a)u(t-a).$$

Solution: This is done on page 386 of the textbook.