VANDERBILT UNIVERSITY
MATH 198 —METHODS OF ORDINARY DIFFERENTIAL EQUATIONS.
PRACTICE FINAL.

The Laplace transform.

The table below indicates the Laplace transform $F(s)$ of the given function $f(t)$.

$f(t)$	$F(s)$
1	$\frac{1}{s}$
t	$\frac{1}{s^2}$
t^n	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
$\cos(kt)$	$\frac{s}{s^2+k^2}$
$\sin(kt)$	$\frac{k}{s^2+k^2}$
$e^{at}\cos(kt)$	$\frac{s-a}{(s-a)^2+k^2}$
$e^{at}\sin(kt)$	$\frac{k}{(s-a)^2+k^2}$

The following are the main properties of the Laplace transform.

Function	Laplace transform
$af(t)+bg(t)$	$aF(s)+bG(s)$
$f'(t)$	$sF(s)-f(0)$
$f''(t)$	$s^2F(s)-sf(0)-f'(0)$
$f^{(n)}(t)$	$s^nF(s)-s^{n-1}f(0)-\cdots-f^{(n-1)}(0)$
$e^{at}f(t)$	$F(s-a)$
$t^nf(t)$	$(-1)^n\frac{d^n}{ds^n}F(s)$
$(f*g)(t)$	$F(s)G(s)$
$u(t-a)$	$\frac{e^{-as}}{s}$

Above, $f*g$ is the convolution of f and g, given by

$$(f*g)(t)=\int_0^t f(t-\tau)g(\tau)\,d\tau,$$

and $u(t-a)$ is given by

$$u(t-a)=\begin{cases}0, & t<a,\\ 1, & t>a.\end{cases}$$

Question 1. Find the general solution of the the differential equations below.

(a) $y' + \dfrac{2xy - 3x^2}{x^2 - 2y^{-3}} = 0.$

(b) $\dfrac{dy}{dx} = 2 - \sqrt{2x - y + 3}.$

(c) $y' - 4y = 32x^2$.

(d) $y' = \dfrac{x}{y} + \dfrac{y}{x}$.

(e) $y'' - 5y' + 6y = 0$.

(f) $y' + \dfrac{y}{x} = -\dfrac{4x}{y^2}$.

Question 2. Solve the following initial value problems.

(a) $y'' + 9y = 10e^{2t}$, $y(0) = -1$, $y'(0) = 5$.

(b) $y'' + 3y' + 4y = u(t-1)$, $y(0) = 0$, $y'(0) = 1$.

(c) $y(t) + \int_0^t y(\tau)(t-\tau)\,d\tau = e^{-3t}$.

Question 3. Using power series, find the general solution to the differential equations below (your solution should include the general form of the coefficients a_n).

(a) $(1-x^2)y'' + xy' + 3y = 0$.

(b) $(x^2 - 2)y'' + 3y = 0$.

Question 4. Find a power series solution to the differential equations below about the given point (your solution should include the general form of the coefficients a_n).

(a) $4x^2y'' + 2x^2y' - (x+3)y = 0$, $x > 0$, about $x = 0$.

(b) $xy'' + (x-1)y' - 2y = 0$, $x > 0$, about $x = 0$.

Question 5. Find at least the first three nonzero terms in the series expansion about $x = 0$ for a general solution of

$$xy'' - y' - xy = 0, \; x > 0.$$

Question 6. Find the general solution of the system

$$x' = Ax$$

for the given matrices A.

(a) $A = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix}$.

(b) $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}.$

Question 7. Let $F(s) = \mathscr{L}\{f\}(s)$ exist for $s > \alpha$, $\alpha \geq 0$. Show that if $a > 0$, then

$$\mathscr{L}^{-1}\{e^{-as}\} = f(t-a)u(t-a).$$