**VANDERBILT UNIVERSITY**
**MATH 198 —METHODS OF ORDINARY DIFFERENTIAL EQUATIONS**
**EXAMPLES OF CHAPTER 9**

**Question 1.** Solve the linear system:

$$\begin{cases} x + 3y + 2z = 2 \\ 2x + 7y + 7z = -1 \\ 2x + 5y + 2z = 7 \end{cases}$$

**Question 2.** Identify a matrix that describes the system of question 1.

**Question 3.** Solve the system of question 1 using matrices.

**Question 4.** Consider the differential equation $y'' - 10y + 21y = 0$. Verify that $y = Ae^{3x} + Be^{7x}$ is a solution, with $A$ and $B$ arbitrary constants. If in addition $y(0) = 15$, $y'(0) = 13$, write a system for $A$ and $B$.

**Question 4.** Find the determinant of

$$A = \begin{bmatrix} 3 & 5 & 6 \\ 2 & 4 & 3 \\ 2 & 3 & 5 \end{bmatrix}$$

**SOLUTIONS.**

**1.** Subtract twice the first equation from the second one and replace the second equation by the result to get

$$\begin{cases} x + 3y + 2z = 2 \\ y + 3z = -5 \\ 2x + 5y + 2z = 7 \end{cases}$$

Subtract twice the first equation from the third one and replace the third equation by the result to get

$$\begin{cases} x + 3y + 2z = 2 \\ y + 3z = -5 \\ -y + -2z = 3 \end{cases}$$

Adding the last two equations

$$\begin{cases} x + 3y + 2z = 2 \\ y + 3z = -5 \\ z = -2 \end{cases}$$

Multiply the third equation by $-3$, add to the second one and replace the second equation with the result to get

$$\begin{cases} x + 3y + 2z = 2 \\ y = 1 \\ z = -2 \end{cases}$$

Multiply the third equation by $-2$, add to the first one to obtain

$$\begin{cases} x + 3y = 6 \\ y = 1 \\ z = -2 \end{cases}$$

Multiply the second equation by $-3$ and add to the first one to finally obtain

$$\begin{cases} x = 3 \\ y = 1 \\ z = -2 \end{cases}$$

So the solution of the system is $x = 3$, $y = 1$, $z = -2$.

**2.** From the original system we read off

$$\begin{bmatrix} 1 & 3 & 2 & \vdots & 2 \\ 2 & 7 & 7 & \vdots & -1 \\ 2 & 5 & 2 & \vdots & 7 \end{bmatrix}$$

**3.** The matrix of the system is

$$\begin{bmatrix} 1 & 3 & 2 & \vdots & 2 \\ 2 & 7 & 7 & \vdots & -1 \\ 2 & 5 & 2 & \vdots & 7 \end{bmatrix}$$

Then

$$\begin{bmatrix} 1 & 3 & 2 & \vdots & 2 \\ 2 & 7 & 7 & \vdots & -1 \\ 2 & 5 & 2 & \vdots & 7 \end{bmatrix} \underset{\sim}{\begin{matrix} L_2 \leftarrow -2L_1 + L_2 \\ L_3 \leftarrow -2L_1 + L_3 \end{matrix}} \begin{bmatrix} 1 & 3 & 2 & \vdots & 2 \\ 0 & 1 & 3 & \vdots & -5 \\ 0 & -1 & -2 & \vdots & 3 \end{bmatrix}$$

$$\underset{\sim}{L_3 \leftarrow L_2 + L_3} \begin{bmatrix} 1 & 3 & 2 & \vdots & 2 \\ 0 & 1 & 3 & \vdots & -5 \\ 0 & 0 & 1 & \vdots & -2 \end{bmatrix} \underset{\sim}{\begin{matrix} L_2 \leftarrow -3L_3 + L_2 \\ L_1 \leftarrow -2L_3 + L_1 \end{matrix}} \begin{bmatrix} 1 & 3 & 0 & \vdots & 6 \\ 0 & 1 & 0 & \vdots & 1 \\ 0 & 0 & 1 & \vdots & -2 \end{bmatrix}$$

$$L_1 \leftarrow \underset{\sim}{-3L_2} + L_1 \quad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

Therefore the solution of the system is $x = 3$, $y = 1$, $z = -2$.

**4.** We have

$$y = Ae^{3x} + Be^{7x},$$

$$y' = 3Ae^{3x} + 7Be^{7x},$$

$$y'' = 9Ae^{3x} + 49Be^{7x}.$$

Then

$$\begin{aligned} y'' - 10y + 21y &= 9Ae^{3x} + 49Be^{7x} - 10(3Ae^{3x} + 7Be^{7x}) + 21(Ae^{3x} + Be^{7x}) \\ &= (9A - 30A + 21A)e^{3x} + (49B - 70B + 21B)e^{7x} \\ &= 0. \end{aligned}$$

Plugging zero into the expression for $y$ and using $y(0) = 15$ we find $A + B = 15$, and plugging zero into the expression for $y'$ and using $y'(0) = 13$ we obtain $3A + 7B = 13$, so

$$\begin{cases} A + B = 15 \\ 3A + 7B = 13 \end{cases}$$

**5.** We have

$$A_{11} = \begin{bmatrix} 4 & 3 \\ 3 & 5 \end{bmatrix} \Rightarrow \det(A_{11}) = 4 \cdot 5 - 3 \cdot 3 = 11,$$

$$A_{12} = \begin{bmatrix} 2 & 3 \\ 2 & 5 \end{bmatrix} \Rightarrow \det(A_{12}) = 2 \cdot 5 - 3 \cdot 2 = 4,$$

$$A_{13} = \begin{bmatrix} 2 & 4 \\ 2 & 3 \end{bmatrix} \Rightarrow \det(A_{13}) = 2 \cdot 3 - 2 \cdot 4 = -2.$$

Hence,

$$\det(A) = 3\det(A_{11}) - 5\det(A_{12}) + 6\det(A_{13}) = 33 - 20 - 12 = 1.$$

*URL*: `http://www.disconzi.net/Teaching/MAT198-Spring-14/MAT198-Spring-14.html`