VANDERBILT UNIVERSITY
MATH 198 —METHODS OF ORDINARY DIFFERENTIAL EQUATIONS
EXAMPLES OF SECTIONS 4.6 AND 4.7.

Question 1. Find the particular solution y_p of the equation

$$x^2y'' - 4xy' + 6y = x^3. \tag{1}$$

Question 2. Show that the formula given in class for y_p, namely,

$$y_p(x) = -y_1(x)\int \frac{y_2(x)f(x)}{W(x)}\,dx + y_2(x)\int \frac{y_2(x)f(x)}{W(x)}\,dx. \tag{2}$$

in fact yields a solution to

$$y''(x) + p(x)y'(x) + q(x)y(x) = f(x). \tag{3}$$

Solutions.

1. First, we need to solve the homogeneous equation

$$x^2y'' - 4xy' + 6y = 0. \tag{4}$$

This is a Cauchy-Euler equation with $a = 1$, $b = -4$ and $c = 6$. Its characteristic equation is then

$$a\lambda^2 + (b-a)\lambda + c = \lambda^2 - 5\lambda + 6 = 0,$$

whose solutions are

$$\lambda_1 = 2, \text{ and } \lambda_2 = 3.$$

Therefore

$$y_1 = x^2, \text{ and } y_2 = x^3$$

are two linearly independent solutions of (4).

To find y_p, we use (2). It is very important to notice that (2) holds when the equation is written as in (3), i.e., with the coefficient of y'' equal to 1. Thus, we divide (1) by x^2, obtaining

$$y'' - \frac{4}{x}y' + \frac{6}{x^2}y = x,$$

in which case $f(x) = x$. We have

$$W = y_1y_2' - y_2y_1' = x^2 3x^2 - x^3 2x = x^4$$

(notice that $W \neq 0$ since, as we recall, we solve the Cauchy-Euler equation for $x > 0$). Then

$$\int \frac{y_2(x)f(x)}{W(x)}\,dx = \int \frac{x^3x}{x^4}\,dx = x,$$

and

$$\int \frac{y_1(x)f(x)}{W(x)}\,dx = \int \frac{x^2x}{x^4}\,dx = \ln x.$$

We thus find

$$y_p = x^3(\ln x - 1).$$

2. We have to plug (2) into (3). Since we shall differentiate y_p, it is useful to remember the Fundamental Theorem of Calculus, which gives

$$\Big(\int \frac{y_2(x)f(x)}{W(x)}\,dx\Big)' = \frac{y_2(x)f(x)}{W(x)},$$

and

$$\Big(\int \frac{y_2(x)f(x)}{W(x)}\,dx\Big)' = \frac{y_2(x)f(x)}{W(x)}.$$

Using these formulas and the product rule we find

$$y_p' = -y_1'\int \frac{y_2 f}{W} - y_1\frac{y_2 f}{W} + y_2'\int \frac{y_1 f}{W} + y_2\frac{y_1 f}{W},$$

where we write $\int \frac{y_2 f}{W}$ instead of $\int \frac{y_2(x)f(x)}{W(x)}\,dx$ in order to simplify the notation (analogously for the other integral). Taking another derivative

$$y_p'' = -y_1''\int \frac{y_2 f}{W} - 2y_1'\frac{y_2 f}{W} - y_1\left(\frac{y_2 f}{W}\right)' + y_2''\int \frac{y_1 f}{W} + 2y_2'\frac{y_1 f}{W} + y_2\left(\frac{y_1 f}{W}\right)'.$$

Using y_p, y_p' and y_p'' into the equation we find

$$\begin{aligned} y_p'' + py_p' + qy_p = &-\big(y_1'' + py_1' + qy_1\big)\int \frac{y_2 f}{W} + \big(y_2'' + py_2' + qy_2\big)\int \frac{y_2 f}{W} \\ &- \frac{y_2 f}{W}(py_1 + 2y_1') + \frac{y_1 f}{W}(py_2 + 2y_2') - y_1\left(\frac{y_2 f}{W}\right)' + y_2\left(\frac{y_1 f}{W}\right)'. \end{aligned}$$

Since by hypothesis y_1 and y_2 are solutions of the homogeneous equation,

$$y_1'' + py_1' + qy_1 = 0,$$

and

$$y_2'' + py_2' + qy_2 = 0,$$

so

$$y_p'' + py_p' + qy_p = -\frac{y_2 f}{W}(py_1 + 2y_1') + \frac{y_1 f}{W}(py_2 + 2y_2') - y_1\left(\frac{y_2 f}{W}\right)' + y_2\left(\frac{y_1 f}{W}\right)'$$

By the product rule

$$\left(\frac{y_2 f}{W}\right)' = y_2'\frac{f}{W} + y_2 f'\frac{1}{W} + y_2 f\left(\frac{1}{W}\right)',$$

and

$$\left(\frac{y_1 f}{W}\right)' = y_1'\frac{f}{W} + y_1 f'\frac{1}{W} + y_1 f\left(\frac{1}{W}\right)'.$$

Therefore

$$\begin{aligned} y_p'' + py_p' + qy_p = &-\frac{y_2 f}{W}(py_1 + 2y_1') + \frac{y_1 f}{W}(py_2 + 2y_2') \\ &- y_1 y_2'\frac{f}{W} - y_1 y_2 f'\frac{1}{W} - y_1 y_2 f\left(\frac{1}{W}\right)' \\ &+ y_2 y_1'\frac{f}{W} + y_2 y_1 f'\frac{1}{W} + y_2 y_1 f\left(\frac{1}{W}\right)'. \end{aligned}$$

Notice that the last two terms of the second line cancel with the last two terms of the third line. We are left with

$$\begin{aligned} y_p'' + p y_p' + q y_p &= -\frac{y_2 f}{W}\big(p y_1 + 2 y_1'\big) + \frac{y_1 f}{W}\big(p y_2 + 2 y_2'\big) - y_1 y_2' \frac{f}{W} + y_2 y_1' \frac{f}{W} \\ &= -p\frac{y_2 f}{W} y_1 - 2\frac{y_2 f}{W} y_1' + p\frac{y_1 f}{W} y_2 + 2\frac{y_1 f}{W} y_2' - y_1 y_2' \frac{f}{W} + y_2 y_1' \frac{f}{W}. \end{aligned}$$

The first and third terms on the last line cancel out, and then

$$\begin{aligned} y_p'' + p y_p' + q y_p &= -2\frac{y_2 f}{W} y_1' + 2\frac{y_1 f}{W} y_2' - y_1 y_2' \frac{f}{W} + y_2 y_1' \frac{f}{W} \\ &= -2\frac{f}{W} y_2 y_1' + 2\frac{f}{W} y_1 y_2' - y_1 y_2' \frac{f}{W} + y_2 y_1' \frac{f}{W} \\ &= \frac{f}{W} y_1 y_2' - \frac{f}{W} y_2 y_1' = \frac{f}{W}\big(y_1 y_2' - y_2 y_1'\big) \\ &= f, \end{aligned}$$

where in the last step we used that $W = y_1 y_2' - y_2 y_1'$.

URL: `http://www.disconzi.net/Teaching/MAT198-Spring-14/MAT198-Spring-14.html`