VANDERBILT UNIVERSITY
MATH 198 —METHODS OF ORDINARY DIFFERENTIAL EQUATIONS
EXAMPLES OF SECTION 1.2.

Question 1. Consider the theorem discussed in class for existence and uniqueness of solutions to the initial value problem

$$\begin{cases} \frac{dy}{dx} = f(x,y), \\ y(a) = b. \end{cases}$$

If the hypotheses of the theorem are satisfied, does it follow that the initial value problem always admits only one solution?

SOLUTIONS. No. Let us consider the following counter-example:

$$\begin{cases} \frac{dy}{dx} = \frac{2y}{x}, \\ y(-1) = 1. \end{cases} \tag{1}$$

In this case $f(x,y) = \frac{2y}{x}$ and we immediately check that

$$\frac{\partial f}{\partial y} = \frac{2}{x},$$

satisfying the hypotheses of the theorem at $(-1,1)$. Therefore, we conclude that the initial value problem (1) has a unique solution on some interval (a,b) containing -1. It is straightforward to check that $y(x) = x^2$ satisfies (1), so it must be, by uniqueness, the solution to the initial value problem in the neighborhood of $(-1,1)$.

However, consider the function $\widetilde{y}$ given by

$$\widetilde{y}(x) = \begin{cases} x^2, & \text{if } x \leq 0, \\ Cx^2, & \text{if } x > 0, \end{cases}$$

where C is a constant different than zero. Plugging in, we can check that $\widetilde{y}$ also satisfies (1), but $\widetilde{y}$ is obviously non-unique, as one obtains a different function for each value of C.

This does not contradict the uniqueness guaranteed by the theorem, because it assures uniqueness only in the neighborhood of the point $(-1,1)$. In fact, $\widetilde{y}$ agrees with $y = x^2$ for $x \leq 0$ regardless of the value of C. What happens is that the unique solution curve near $(-1,1)$ branches at the origin into infinitely many solutions.

URL: `http://www.disconzi.net/Teaching/MAT198-Spring-14/MAT198-Spring-14.html`