VANDERBILT UNIVERSITY
MATH 196 — SOLVING $x' = Ax$

Let A be an $n \times n$ matrix with constant entries. To solve

$$x' = Ax \tag{1}$$

proceed as follows.

I. Compute the characteristic polynomial and find its roots, i.e., solve

$$\det(A - \lambda I) = 0 \tag{2}$$

for λ, where I is the identity matrix. Equation (2) is a polynomial of degree n in λ. It will have n (possibly complex) roots counted with multiplicity.

II. For each eigenvalue λ with multiplicity m (possibly with $m = 1$), find u solving

$$(A - \lambda I)u = 0. \tag{3}$$

Try to obtain m linearly independent solutions $u_1, \dots, u_m$ to (3), i.e., m linearly independent eigenvectors associated with λ. The number of linearly independent eigenvectors will be equal to the number of free variables in the reduced row echelon form of the system. If this is not the case, i.e., if (3) has less than m linearly independent eigenvectors, solve

$$(A - \lambda I)^m u = 0, \tag{4}$$

finding m linearly independent solutions, i.e., m linearly independent generalized eigenvectors.

III. For each eigenvalue λ with corresponding generalized eigenvectors $u_1, \dots, u_m$, form m linearly independent solutions of (1) by

$$x_i = e^{\lambda t}\Big[u_i + t(A - \lambda I)u_i + \frac{t^2}{2!}(A - \lambda I)^2 u_i + \cdots + \frac{t^{m-1}}{(m-1)!}(A - \lambda I)^{m-1} u_i\Big],\ i = 1, \dots, m.$$

Notice that the above simplifies to $x_i = e^{\lambda t} u_i$ when u_i is an eigenvector.

Remark. For complex eigenvalues, do not forget to write real solutions using $e^{i\theta} = \cos\theta + i\sin\theta$.

Example. Solve $x' = Ax$, where

$$A = \begin{bmatrix} 5 & -4 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 5 \end{bmatrix}.$$

A simple computation gives

$$\det \begin{bmatrix} 5-\lambda & -4 & 0 \\ 1 & -\lambda & 2 \\ 0 & 2 & 5-\lambda \end{bmatrix} = -\lambda(\lambda - 5)^2,$$

so $\lambda_1 = 0$ and $\lambda_2 = 5$ are the eigenvalues, with λ_2 of multiplicity two.

To find an eigenvector associated with λ_1, we solve

$$\left[\begin{array}{ccc|c} 5 & -4 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 2 & 5 & 0 \end{array}\right].$$

Applying Gauss-Jordan elimination we find $u_1 = (-4, -5, 2)$, and $x_1 = e^{0t}u_1 = (-4, -5, 2)$ is a solution to $x' = Ax$.

Next, we move to λ_2, and consider:

$$\left[\begin{array}{ccc|c} 0 & -4 & 0 & 0 \\ 1 & -5 & 2 & 0 \\ 0 & 2 & 0 & 0 \end{array}\right].$$

Applying Gauss-Jordan elimination, we find

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Thus, this system has only one free variable, yielding only one linearly independent eigenvector which we can take to be $u_2 = (-2, 0, 1)$. Hence $x_2 = e^{5t}(-2, 0, 1)$ is a second linearly independent solution to $x' = Ax$. To find a third linearly independent solution, we need to find a generalized eigenvector associated with $\lambda_2 = 5$. Compute

$$(A - 5I)^2 = \begin{bmatrix} 0 & -4 & 0 \\ 1 & -5 & 2 \\ 0 & 2 & 0 \end{bmatrix}^2 = \begin{bmatrix} -4 & 20 & -8 \\ -5 & 25 & -10 \\ 2 & -10 & 4 \end{bmatrix}.$$

Now we solve

$$\left[\begin{array}{ccc|c} -4 & 20 & -8 & 0 \\ -5 & 25 & -10 & 0 \\ 2 & -10 & 4 & 0 \end{array}\right].$$

Applying Gauss-Jordan elimination gives

$$\left[\begin{array}{ccc|c} -1 & 5 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

which has two free variables that yield two linearly independent generalized eigenvectors $u_2 = (-2, 0, 1)$ and $u_3 = (5, 1, 0)$ (notice that we already knew from above that u_2 is a solution since it is an eigenvector). To find a third (linearly independent) solution to $x' = Ax$, compute

$$x_3 = e^{5t}(u_3 + t(A - 5I)u_3) = e^{5t}\begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} + te^{5t}\begin{bmatrix} 0 & -4 & 0 \\ 1 & -5 & 2 \\ 0 & 2 & 0 \end{bmatrix}\begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} = e^{5t}\begin{bmatrix} 5 - 4t \\ 1 \\ 2t \end{bmatrix}.$$

The general solution is given by $x = c_1x_1 + c_2x_2 + c_3x_3$.

URL: `http://www.disconzi.net/Teaching/MAT196-Spring-15/MAT196-Spring-15.html`