VANDERBILT UNIVERSITY
MATH 196 — SOLUTIONS TO PRACTICE TEST 2

Question 1. Determine whether or not the given vectors form a basis of $\mathbb{R}^n$.
(a) $v_1 = (3, -1, 2)$, $v_2 = (6, -2, 4)$, $v_3 = (5, 3, -1)$.
(b) $v_1 = (3, -7, 5, 2)$, $v_2 = (1, -1, 3, 4)$, $v_3 = (7, 11, 3, 13)$.
(c) $v_3 = (1, 0, 0, 0)$, $v_2 = (0, 3, 0, 0)$, $v_3 = (0, 0, 7, 6)$, $v_4 = (0, 0, 4, 5)$.

Solution. a) Compute

$$\det[v_1\, v_2\, v_3] = \begin{bmatrix} 3 & 5 & 5 \\ -1 & -2 & 3 \\ 2 & 4 & -1 \end{bmatrix} = 0.$$

The vectors are linearly dependent, hence do not form a basis.
b) A basis of $\mathbb{R}^4$ must contain four vectors, so this is not a basis.
c) Compute

$$\det[v_1\, v_2\, v_3\, v_4] = 66,$$

hence we have four linearly independent vectors in $\mathbb{R}^4$ and therefore a basis.

Question 2. Consider the set W of all vectors $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ such that $x_1 = x_2 + x_3 + x_4$. Is W a sub-space of $\mathbb{R}^4$? In case yes, find a basis for W.

Solution. Any such vector can be written as $x_2(1, 1, 0, 0) + x_3(1, 0, 1, 0) + x_4(1, 0, 0, 1)$, which clearly forms a subspace with $(1, 1, 0, 0)$, $(1, 0, 1, 0)$ $(1, 0, 0, 1)$ as basis.

Question 3. Find a basis for the solution space of the linear system

$$\begin{cases} x_1 + 3x_2 - 4x_3 - 8x_4 + 6x_5 = 0 \\ x_1 \qquad\quad + 2x_3 + x_4 + x_5 = 0 \\ 2x_1 + 7x_2 - 10x_3 - 19x_4 + 13x_5 = 0 \end{cases}$$

Solution. The rref of the matrix of the system is

$$\begin{bmatrix} 1 & 0 & 2 & 1 & 0 \\ 0 & 1 & -2 & -3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

A basis is then given by $(-2, 2, 1, 0, 0)$, and $(-1, 3, 0, 1, 0)$.

Question 4. Let $\{v_1, v_2, \ldots, v_n\}$ be a basis of $\mathbb{R}^n$, and let A be an invertible $n \times n$ matrix. Consider the vectors $u_1 = Av_1$, $u_2 = Av_2$, $\ldots$, $u_n = Av_n$. Prove that $\{u_1, u_2, \ldots, u_n\}$ is also a basis of $\mathbb{R}^n$.

Solution. Since the set $\{u_1, u_2, \ldots, u_n\}$ contains n vectors, it suffices to show that they are linearly independent. Let $c_1, \ldots, c_n$ be such that

$$c_1u_1 + c_2u_2 + \cdots c_nu_n = 0,$$

we have to show that all the constants are equal to zero. But

$$c_1u_1 + c_2u_2 + \cdots c_nu_n = c_1Av_1 + c_2Av_2 + \cdots c_nAv_n$$

$$= A(c_1v_1 + c_2v_2 + \cdots c_nv_n)$$
$$= 0.$$

Denote by $w = c_1v_1 + c_2v_2 + \cdots c_nv_n$, so we have

$$Aw = 0.$$

But because A is invertible, we conclude that $w = 0$, i.e.

$$c_1v_1 + c_2v_2 + \cdots c_nv_n = 0.$$

Invoking that $\{v_1, v_2, \dots, v_n\}$ is a basis of $\mathbb{R}^n$, we obtain that $c_1 = c_2 = \cdots = c_n = 0$, as desired.

Question 5. Let u and v be arbitrary vectors in a vector space V. Recall that the norm or length of a vector is defined by $\| v \|= \sqrt{\langle v, v\rangle}$, where $\langle\, ,\, \rangle$ is denotes an inner product on V. Show that

(a)

$$\| u + v \|^2 + \| u - v \|^2= 2 \| u \|^2 +2 \| v \|^2 .$$

(b)

$$\| u + v \|^2 - \| u - v \|^2= 4\langle u, v\rangle.$$

Solution. Direct computation.

Question 6. Let $S = \{u_1, u_2\}$ and $T = \{v_1, v_2\}$ be linearly independent sets of vectors such that each u_i in S is orthogonal to every vector v_j in T. Shot that u_1, u_2, v_1, v_2 are linearly independent.

Solution. Consider $\{u_1, u_2, v_1\}$. We claim that this set is linearly independent. Since u_1 and u_2 are linearly independent by hypothesis, it suffices to show that v_1 does not belong to span$\{u_1, u_2\}$. If

$$c_1u_1 + c_2u_2 = v_1$$

then, taking inner product with v_1 we obtain

$$\| v_1 \|^2= 0,$$

which implies that $v_1 = 0$ — but this cannot be the case since T is linearly independent and therefore it does not contain the zero vector. Analogously we obtain that v_2 cannot be spanned by u_1 and u_2. A similar argument finally shows that $\{u_1, u_2, v_1, v_2\}$ is linearly independent.

Question 7. Let W be a subspace of $\mathbb{R}^n$. Prove that $W^\perp$ is also a subspace. If the dimension of W is d, what is the dimension of $W^\perp$?

Solution. Obviously $0 \in W^\perp$. If $u, v \in W^\perp$, then, for any $w \in W$, we have $\langle u + v, w\rangle = \langle u, w\rangle + \langle v, w\rangle = 0 + 0 = 0$. Thus $u + v \in W^\perp$. Similarly $cu \in W^\perp$ for any $c \in \mathbb{R}$ if $u \in W^\perp$. Therefore, $W^\perp$ is a subspace. If W is d-dimensional, $W^\perp$ is $n - d$-dimensional.

Question 8. Let W_1 and W_2 be two subspaces of a vector space V. Show that $W_1 \cap W_2$ is also a subspace of V.

Solution. Clearly $0 \in W_1 \cap W_2$. If u and v belong to $W_1 \cap W_2$, then they belong simultaneously to W_1 and W_2. Since $u, v \in W_1$, and W_1 is a subspace, it follows that $u + v \in W_1$. Similarly $u + v \in W_2$. Therefore $u+v$ belongs to both W_1 and W_2, and thus to their intersection. Analogously one shows that $cu \in W_1 \cap W_2$ if $u \in W_1 \cap W_2$ and $c \in \mathbb{R}$.

Question 9. Find a basis for the span of the following set of vectors, and determine its dimension.

(a) The polynomials 2, x, $2x - 3$, $2x^3 + 1$.

(b) The matrices

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

Solution. (a) The span is the set of polynomials of the form $ax^3 + bx + c$. A basis is given by the polynomials 2, x, and $2x^3 + 1$.

(b) Notice that the third matrix is equal to the first minus the second, while the first two matrices are clearly linearly independent. Thus the span is the set of matrices of the form

$$\begin{bmatrix} a+b & -b \\ -b & a \end{bmatrix},$$

and the first two matrices give a basis.

Question 10. Consider the set of all continuous functions $f : [0,1] \to \mathbb{R}$. Is it a vector space? In case yes, what is its dimension?

Solution. It was shown in class that this is a vector space. All the monomials $1, x, x^2, x^3, x^4, \dots$ belong to this vector space. Since there are infinitely many of them, and they are all linearly independent because any two different powers of x are linearly independent, this space is infinite dimensional.

Question 11. True or false? Justify your answer.

(a) Let A be a square matrix. If the system $A\vec{x} = \vec{b}$ always has a solution for any vector $\vec{b}$, then the determinant of A is zero.

(b) The set of all 3×3 invertible matrices is a subspace of the vector space of all 3×3 matrices.

(c) Let A be a $n \times m$ matrix. Suppose that there exists a vector $\vec{b} \in \mathbb{R}^n$ such that $A\vec{x} = \vec{b}$ has no solution. Then the rank of A is less than n.

(d) If A is $n \times m$, and B is $m \times \ell$, then the product AB is well defined.

(e) Let A be a $n \times m$ matrix and $\vec{b} \in \mathbb{R}^n$. The set of all vectors $\vec{x} \in \mathbb{R}^m$ that solve the system $A\vec{x} = \vec{b}$ is a subspace of $\mathbb{R}^m$ if, and only if, $\vec{b} = \vec{0}$. In particular, if $\vec{b} \neq \vec{0}$, then set of all vectors $\vec{x} \in \mathbb{R}^m$ that solve the system $A\vec{x} = \vec{b}$ is never a subspace of $\mathbb{R}^m$.

Solution. (a) F (b) F (c) T (d) T (e) T.

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