VANDERBILT UNIVERSITY
MATH 196 — PRACTICE TEST 2

Question 1. Determine whether or not the given vectors form a basis of $\mathbb{R}^n$.
(a) $v_1 = (3, -1, 2)$, $v_2 = (6, -2, 4)$, $v_3 = (5, 3, -1)$.
(b) $v_2 = (3, -7, 5, 2)$, $v_2 = (1, -1, 3, 4)$, $v_3 = (7, 11, 3, 13)$.
(c) $v_3 = (1, 0, 0, 0)$, $v_2 = (0, 3, 0, 0)$, $v_3 = (0, 0, 7, 6)$, $v_4 = (0, 0, 4, 5)$.

Question 2. Consider the set W of all vectors $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ such that $x_1 = x_2 + x_3 + x_4$. Is W a sub-space of $\mathbb{R}^4$? In case yes, find a basis for W.

Question 3. Find a basis for the solution space of the linear system

$$\begin{cases} x_1 + 3x_2 - 4x_3 - 8x_4 + 6x_5 = 0 \\ x_1 + 2x_3 + x_4 + x_5 = 0 \\ 2x_1 + 7x_2 - 10x_3 - 19x_4 + 13x_5 = 0 \end{cases}$$

Question 4. Let $\{v_1, v_2, \ldots, v_n\}$ be a basis of $\mathbb{R}^n$, and let A be an invertible $n \times n$ matrix. Consider the vectors $u_1 = Av_1$, $u_2 = Av_2$, $\ldots$, $u_n = Av_n$. Prove that $\{u_1, u_2, \ldots, u_n\}$ is also a basis of $\mathbb{R}^n$.

Question 5. Let u and v be arbitrary vectors in a vector space V. Recall that the norm or length of a vector is defined by $\| v \| = \sqrt{\langle v, v \rangle}$, where $\langle \, , \, \rangle$ is denotes an inner product on V. Show that
(a)

$$\| u + v \|^2 + \| u - v \|^2 = 2 \| u \|^2 + 2 \| v \|^2 .$$

(b)

$$\| u + v \|^2 - \| u - v \|^2 = 4 \langle u, v \rangle .$$

Question 6. Let $S = \{u_1, u_2\}$ and $T = \{v_1, v_2\}$ be linearly independent sets of vectors such that each u_i in S is orthogonal to every vector v_j in T. Shot that u_1, u_2, v_1, v_2 are linearly independent.

Question 7. Let W be a subspace of $\mathbb{R}^n$. Prove that $W^\perp$ is also a subspace. If the dimension of W is d, what is the dimension of $W^\perp$?

Question 8. Let W_1 and W_2 be two subspaces of a vector space V. Show that $W_1 \cap W_2$ is also a subspace of V.

Question 9. Find a basis for the span of the following set of vectors, and determine its dimension.
(a) The polynomials 2, x, $2x - 3$, $2x^3 + 1$.
(b) The matrices

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

Question 10. Consider the set of all continuous functions $f : [0, 1] \to \mathbb{R}$. Is it a vector space? In case yes, what is its dimension?

Question 11. True or false? Justify your answer.

(a) Let A be a square matrix. If the system $A\vec{x} = \vec{b}$ always has a solution for any vector $\vec{b}$, then the determinant of A is zero.

(b) The set of all 3×3 invertible matrices is a subspace of the vector space of all 3×3 matrices.

(c) Let A be a $n \times m$ matrix. Suppose that there exists a vector $\vec{b} \in \mathbb{R}^n$ such that $A\vec{x} = \vec{b}$ has no solution. Then the rank of A is less than n.

(d) If A is $n \times m$, and B is $m \times \ell$, then the product AB is well defined.

(e) Let A be a $n \times m$ matrix and $\vec{b} \in \mathbb{R}^n$. The set of all vectors $\vec{x} \in \mathbb{R}^m$ that solve the system $A\vec{x} = \vec{b}$ is a subspace of $\mathbb{R}^m$ if, and only if, $\vec{b} = \vec{0}$. In particular, if $\vec{b} \neq \vec{0}$, then set of all vectors $\vec{x} \in \mathbb{R}^m$ that solve the system $A\vec{x} = \vec{b}$ is never a subspace of $\mathbb{R}^m$.

Question 12.

(a) Know the precise definitions of linear combination, linear dependence, linear independence, basis, vector spaces, and subspaces.

(b) Given a matrix A, know how to find its column space, row space, and its kernel. Understand the relations between these spaces.

(c) Understand the examples posted on the course webpage.

URL: `http://www.disconzi.net/Teaching/MAT196-Spring-15/MAT196-Spring-15.html`