VANDERBILT UNIVERSITY
MATH 196 — SOLUTIONS TO PRACTICE TEST 1

Question 1. Classify the differential equations below as linear or non-linear and state their order.
(a) $y' + y^2 = 0$
(b) $\frac{d^2x}{dt^2} + 25x = \cos(t)$
(c) $yy'' = \sqrt{y}$
(d) $e^{\sin x^2}\frac{dy}{dx} + xy = e^{-x}$
(e) $e^{\cos x^4}\frac{dy}{dx}y = e^{-x}$

Solution.
(a) Non-linear first order.
(b) Linear second order.
(c) Non-linear second order.
(d) Linear first-order.
(e) Non-linear first order.

Question 2. The acceleration of an object moving in a straight line is proportional to the logarithm of the time elapsed since its departure. Find an equation for its position after time t. Is this a well defined problem?

Solution. Write

$$\frac{dv}{dt} = k \ln t \Rightarrow \int dv = k \int \ln t \, dt,$$

so

$$v(t) = kt \ln t - kt + v_0.$$

Notice that this is defined at $t = 0$ since $t \ln t \to 0$ as $t \to 0^+$. Integrating again gives

$$x(t) = -\frac{3}{4}kt^2 + \frac{1}{2}kt^2 \ln t + v_0 t + x_0,$$

where again we see that this is well-defined at $t = 0$.

Question 3. A $300\,\ell$ tank initially contains $10\,kg$ of salt dissolved in $100\,\ell$ of water. Brine containing $2\,kg/\ell$ of salt flows into the tank at the rate $4\,\ell/\min$, and the well-stirred mixture flows out of the tank at the rate $2\,\ell/\min$. How much salt does the tank contain when 80% of its capacity is full?

Solution. Let $x(t)$ and $V(t)$ be respectively the amount of salt in the tank and the volume at time t. Then

$$\frac{dx}{dt} = in - out = 2\,kg/\ell \times 4\,\ell/\min - \frac{x(t)\,kg}{V(t)\,\ell} \times 2\,\ell/\min.$$

The volume at time t is $V(t) = 100 + 4t - 2t = 100 + 2t$, so

$$\frac{dx}{dt} + \frac{x}{50+t} = 8.$$

This is a linear first order equation, where the initial condition is $x(0) = 10$. Using the formula derived in class we find

$$x(t) = \frac{4(t^2 + 100t + 125)}{50 + t}.$$

The tank will be 80 % full when $V(t) = 100 + 2t = 240$, so $t = 70$. Then $x(70) = 601.25\, kg$.

Question 4. Solve the following differential equations:
(a) $y' = -\frac{2xy^3 + e^x}{3x^2y^2 + \sin y}$
(b) $-x^2y' + xy^2 + 3y^2 = 0$
(c) $x^2y' = xy + y^2$
(d) $x^3 + 3y - xy' = 0.$
(e) $y' = x^2 - 2xy + y^2$

Solution.

(a). This is an exact equation. Using the methods of section 1.6 we find the (implicit) solution $x^2y^3 + e^x - \cos y = C$.

(b) This is a separable equation. Separating variables and integrating we find $y = \frac{x}{3 - x \ln x - Cx}$.

(c) This is a homogeneous equation. Using the methods of section 1.6 we find the solution $y = \frac{x}{C - \ln x}$.

(d) Writing the equation as $y' - \frac{3}{x}y = x^3$ we obtain that this is a linear equation. Using the formula for linear equations derived in class (also in page 49 of the textbook) we find $y = x^3(\ln x + C)$.

(e) Making the substitution $v = y - x$ we obtain $v' = v^2$, which is a separable equation for v. Solving and returning to y gives the (implicit) solution $y - x - 1 = Ce^{2x}(y - x + 1)$.

Question 5. Consider the differential equation:

$$y'' + p(x)y' + q(x)y = 0,$$

and suppose that y_1 and y_2 are two solutions. Let c_1 and c_2 be two arbitrary constants. Show that $y = c_1y_1 + c_2y_2$ solves the equation.

Solution. It follows by plugging y into the equation.

Question 6. Solve the linear systems below, when possible.
(a)

$$\begin{cases} 3x + 5y - z = 13 \\ 2x + 7y + z = 28 \\ x + 7y + 2z = 32 \end{cases}$$

Solution. The augmented matrix of the system is

$$\begin{bmatrix} 3 & 5 & -1 & \vdots & 13 \\ 2 & 7 & 1 & \vdots & 18 \\ 1 & 7 & 2 & \vdots & 32 \end{bmatrix}$$

Applying Gauss-Jordan elimination we find

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

So that $x = 1$, $y = 3$, $z = 5$.

(b)

$$\begin{cases} 2x + 3y + 7z = 15 \\ x + 4y + z = 20 \\ x + 2y + 3z = 10 \end{cases}$$

Solution. The augmented matrix of the system is

$$\left[\begin{array}{ccc|c} 2 & 3 & 7 & 15 \\ 1 & 4 & 1 & 20 \\ 1 & 2 & 3 & 10 \end{array}\right]$$

Applying Gauss-Jordan elimination we find

$$\left[\begin{array}{ccc|c} 1 & 0 & 5 & 0 \\ 0 & 1 & -1 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore z is a free variable, and solutions are given by $x = -5z$, $y = 5 + z$, and z can be any real number.

(c)

$$\begin{cases} x - 3y + 2z = 6 \\ x + 4y - z = 4 \\ 5x + 6y + z = 20 \end{cases}$$

Solution. The augmented matrix of the system is

$$\left[\begin{array}{ccc|c} 1 & -3 & 2 & 6 \\ 1 & 4 & -1 & 4 \\ 5 & 6 & 1 & 20 \end{array}\right]$$

Applying Gauss-Jordan elimination we find

$$\left[\begin{array}{ccc|c} 1 & -3 & 2 & 6 \\ 0 & 7 & -3 & -2 \\ 0 & 0 & 0 & -\frac{4}{3} \end{array}\right]$$

The last row means $0 = -\frac{4}{3}$, hence the system is inconsistent, i.e., it has no solution.

Question 7. Let

$$A = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$$

and

$$B = \begin{bmatrix} -1 & 0 & 4 \\ 3 & -2 & 5 \end{bmatrix}.$$

Calculate whichever of the matrices AB and BA is defined.

Solution. AB is well defined, but BA is not. Computing

$$A = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix},$$

$$A = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ -2 \end{bmatrix} = \begin{bmatrix} -2 \\ -6 \end{bmatrix},$$

and

$$A = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 13 \\ 31 \end{bmatrix},$$

we find

$$AB = \begin{bmatrix} 1 & -2 & 13 \\ 5 & -6 & 31 \end{bmatrix}.$$

Question 8. Let

$$A = \begin{bmatrix} 2 & 0 & 0 & -3 \\ 0 & 1 & 11 & 12 \\ 0 & 0 & 5 & 13 \\ -4 & 0 & 0 & 7 \end{bmatrix}$$

What can you say about A^{-1}?

Solution. Applying Gauss-Jordan, one immediately sees that A^{-1} exists.

URL: `http://www.disconzi.net/Teaching/MAT196-Spring-15/MAT196-Spring-15.html`