VANDERBILT UNIVERSITY
MATH 196 — DIFFERENTIAL EQUATIONS WITH LINEAR ALGEBRA
PRACTICE FINAL EXAM.

FORMULAS — These will be given in the final exam.

The table below indicates the Laplace transform $F(s)$ of the given function $f(t)$.

$f(t)$	$F(s)$
1	$\frac{1}{s}$
t	$\frac{1}{s^2}$
t^n	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
$\cos(kt)$	$\frac{s}{s^2+k^2}$
$\sin(kt)$	$\frac{k}{s^2+k^2}$
$e^{at}\cos(kt)$	$\frac{s-a}{(s-a)^2+k^2}$
$e^{at}\sin(kt)$	$\frac{k}{(s-a)^2+k^2}$

The following are the main properties of the Laplace transform.

$af(t)+bg(t)$	$aF(s)+bG(s)$
$f'(t)$	$sF(s)-f(0)$
f''	$s^2F(s)-sf(0)-f'(0)$
$f^{(n)}(t)$	$s^nF(s)-s^{n-1}f(0)-\cdots-f^{(n-1)}(0)$
$(f*g)(t)$	$F(s)G(s)$

The particular solution of

$$y''+p(t)y'+q(t)y=f(t)$$

is

$$y_p(t)=-y_1(t)\int\frac{y_2(t)f(t)}{W(t)}\,dt+y_2(t)\int\frac{y_1(t)f(t)}{W(t)}\,dt,$$

where $y_1(t)$ and $y_2(t)$ are linearly independent solutions of the associated homogeneous problem and $W(t)$ is the Wronskian of $y_1(t)$ and $y_2(t)$.

Question 1. Consider the matrices

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 0 & 5 \\ 3 & 2 & 4 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 6 \\ 3 & 9 \end{bmatrix}.$$

Their reduced row echelon forms, denoted $\mathrm{rref}(A)$ and $\mathrm{rref}(B)$, respectively, are

$$\mathrm{rref}(A) = \begin{bmatrix} 1 & 0 & 2 & -3 \\ 0 & 1 & -1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathrm{rref}(B) = \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}.$$

(a) Find $\mathrm{Ker}(A)$ and $\mathrm{Ker}(B)$ (i.e, the kernels of A and B).
$\mathrm{span}\{(-2,1,1,0),(3,-4,0,1)\}$ and $\mathrm{span}\{(-3,1)\}$.
(b) Find basis for $\mathrm{Col}(A)$ and $\mathrm{Col}(B)$ (i.e., basis for the space of columns of A and B, respectively).
$\{(1,1,3),(1,2,2)\}$ and $\{(2,3)\}$.
(c) Find basis for $\mathrm{Row}(A)$ and $\mathrm{Row}(B)$
$\{(1,0,2,-3),(0,1,-1,4)\}$ and $\{(1,3)\}$.

Question 2. Let

$$A = \begin{bmatrix} 3 & -3 & 1 \\ 2 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix},$$

$$B = \begin{bmatrix} 6 & -5 & 2 \\ 4 & -3 & 2 \\ 2 & -2 & 3 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

Find the general solution of the systems of differential equations:

(a)

$$x' = Ax.$$

$\lambda_1 = 0$, $\lambda_2 = \lambda_3 = 1$, $u_1 = (1,1,0)$, $u_2 = (-1,0,2)$, $u_3 = (3,2,0)$, $x_1 = (1,1,0)$, $x_2 = e^t(-1,0,2)$, $x_3 = e^t(3,2,0)$.

(b)

$$x' = Bx.$$

$\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3, u_1 = (1,1,0), u_2 = (-1,0,2), u_3 = (1,1,1), x_1 = e^t(1,1,0), x_2 = e^{2t}(-1,0,2)$, $x_3 = e^{3t}(1,1,1)$.

(c)

$$x' = Cx.$$

$\lambda_1 = \lambda_2 = 1$, $\lambda_3 = \lambda_4 = 2$. Eigenvector associated to λ_1: $u_1 = (1,0,0,0)$. Eigenvector associated to λ_3: $u_3 = (1,1,1,0)$. $x_1 = e^t(1,0,0,0)$, $x_3 = e^{2t}(1,1,1,0)$.

$$(A - \lambda_1 I)^2 = \begin{bmatrix} 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Generalized eigenvector associated to λ_1: $u_2 = (0, 1, 0, 0)$. $x_2 = e^t(t, 1, 0, 0)$.

$$(A - \lambda_3 I)^2 = \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Generalized eigenvector associated to λ_3: $u_2 = (0, 0, 0, 1)$. $x_4 = e^{2t}(t, t, t, 1)$.

Question 3. Find the general solution of the differential equations below. In the cases involving a particular solution, you do not have to find the specific values of the constants.

(a)

$$(1+x^2)y' + 3xy - 6x = 0.$$

Linear first order equation. $y = 2 + C(x^2+1)^{-\frac{3}{2}}$.

(b)

$$2xyy' - 3y^2 = 4x^2.$$

Homogeneous equation. $y^2 + 4x^2 = Cx^2$.

(c)

$$y'''' - 2y'' + y = e^x + 1 + x^2 \cos x.$$

$y = c_1 e^x + c_2 x e^x + c_3 e^{-x} + c_4 x e^{-x} + (A + Bx + Cx^2)\cos x + (D + Ex + Fx^2)\sin x + G + Hx^2 e^x$.

(d)

$$y''' + 9y' = x \sin x + x^2 e^{2x}.$$

$y = c_1 + c_2 \cos 3x + c_3 \sin 3x + (A + Bx)\cos x + (C + Dx)\sin x + (E + Fx + Gx^2)e^{2x}$.

Question 4. Consider the system of two blocks and three springs shown in the figure below. Notice that the outermost endpoints of springs one and three are attached to walls. Write a system of differential equations that models the dynamics of system (disregard friction).

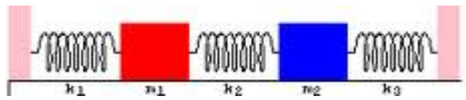

FIGURE 1. Mass-spring system of question 4.

Let x_1 and x_2 be the displacement of blocks 1 and 2 from their respective equilibrium positions. Then

$$\begin{cases} m_1 x_1'' = -(k_1 + k_2)x_1 + k_2 x_2, \\ m_2 x_2'' = k_2 x_1 - (k_2 + k_3)x_2. \end{cases}$$

Question 5. Use Laplace transforms to solve the initial value problems below.
(a)

$$\begin{cases} x'' - 6x' + 8x = 2, \\ x(0) = x'(0) = 0. \end{cases}$$

Taking the Laplace transform, we find $s^2X(s) - 6sX(s) + 8X(s) = \frac{2}{s}$. Then

$$X(s) = \frac{2}{s(s^2 - 6s + 8)} = \frac{1}{4}(\frac{1}{s} + \frac{1}{s-4} - \frac{2}{s-2}).$$

Then $x(t) = \frac{1}{4}(1 + e^{4t} - 2e^{2t})$.
(b)

$$\begin{cases} x'' - 4x = 3t, \\ x(0) = x'(0) = 0. \end{cases}$$

Taking the Laplace transform, we find $s^2X(s) - 4X(s) = \frac{3}{s^2}$. Then

$$X(s) = \frac{3}{s^2(s^2 - 4)} = \frac{3}{4}(\frac{1}{s^2 - 4} - \frac{1}{s^2}).$$

Then $x(t) = \frac{3}{8}\sinh 2t - \frac{3}{4}t$.

Question 6. Let A be an $n \times n$ matrix with real entries. Recall that the transpose of A, denoted A^T, is the matrix obtained from A by the rule:

If the i, j^{th} entries of A are denoted by a_{ij}, then the i, j^{th} entries of A^T are given by a_{ji}.

In other words, A^T is obtained by "switching the rows and columns of A".

Prove that the $\text{Col}(A)$ is orthogonal to $\text{Ker}(A^T)$.

Let $x \in \text{Ker}(A^T)$, so $A^T x = 0$. Then, for any y:

$$0 = \langle A^T x, y \rangle = \langle x, Ay \rangle.$$

Since any element in $\text{Col}(A)$ can be written as Ay for some y, this shows the claim.

Question 7. True or false? Justify your answer.

(a) Let A be a $n \times n$ matrix with real entries, and suppose all its eigenvalues are complex. Because for each eigenvalue λ there are two linearly independent real solutions, we can conclude that there exist $2n$ linearly independent solutions of $x' = Ax$.

(b) If a square matrix A has an eigenvalue λ of multiplicity m, where $m > 1$, then in order to solve $x' = Ax$ we must find vectors that are generalized eigenvectors of A but that are not eigenvectors of A.

(c) Any differential equation of order n is can be written as a $n \times n$ system of first order differential equations.

(d) A $n \times n$ matrix that has n distinct real eigenvalues necessarily has n linearly independent eigenvectors.

(a) F, (b) F, (c) T, (d) T.

Question 8. State the following definitions.

(a) Eigenvalue.
(b) Eigenvector.
(c) Generalized eigenvector.
(d) Defective eigenvalue.