VANDERBILT UNIVERSITY
MATH 196 — EXAMPLES OF SECTION 7.5

Here we will look only at the case of eigenvalues with multiplicity two and defect one. Recall that this means that we have an eigenvalue λ which is a double root of the characteristic equation

$$\det(A-\lambda I)=0,$$

but such that, when we attempt to solve

$$(A-\lambda I)\vec{v}=\vec{0}, \tag{1}$$

we find only one linearly independent eigenvector. Since the multiplicity of λ is two, we need two linearly independent eigenvectors associated wtih λ. Recall that for this to be the case, the system should have two free variable.

When we have such a missing eigenvector, we proceed as follows. Let $\vec{v}_1$ be an eigenvector that we found solving (1). Then find a solution $\vec{v}_2$ of

$$(A-\lambda I)^2\,\vec{v}_2=\vec{0}, \tag{2}$$

satisfying

$$(A-\lambda I)\vec{v}_2=\vec{v}_1. \tag{3}$$

Notice that (3) may not be automatically satisfied, in which case we have to choose the free variables of (2) appropriately.

Then

$$\vec{x}_1=\vec{v}_1e^{\lambda t}$$

and

$$\vec{x}_2=(t\vec{v}_1+\vec{v}_2)e^{\lambda t}$$

are two linearly independent solutions associated with λ.

As an example, consider

$$\vec{x}'=\begin{bmatrix}1 & -3\\ 3 & 7\end{bmatrix}\vec{x}.$$

The characteristic equation is

$$\det\begin{bmatrix}1-\lambda & -3\\ 3 & 7-\lambda\end{bmatrix}=(1-\lambda)(7-\lambda)+9=\lambda^2-8\lambda+16=(\lambda-4)^2=0.$$

The solution is $\lambda=4$, counted with multiplicity 2. Let us find the corresponding eigenvectors.

$$A-\lambda I=\begin{bmatrix}3-\lambda & -3\\ 3 & 7-\lambda\end{bmatrix}=\begin{bmatrix}-3 & -3\\ 3 & 3\end{bmatrix},$$

hence we want to solve

$$\begin{bmatrix}-3 & -3\\ 3 & 3\end{bmatrix}\vec{v}_1=\begin{bmatrix}-3 & -3\\ 3 & 3\end{bmatrix}\begin{bmatrix}a\\ b\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}.$$

We see that the first and second equations, $-3a - 3b = 0$ and $3a + 3b = 0$, are multiples of each other, so we have only one free variable. We find

$$\vec{v}_1 = a \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Dropping a (or, more precisely, choosing $a = 1$),

$$\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Next, compute

$$(A - \lambda I)^2 = \begin{bmatrix} -3 & -3 \\ 3 & 3 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

and solve (2), i.e.,

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Any values of a and b solve this system, so we could choose $a = 1$, $b = 0$ and set

$$\vec{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

However, recall that we also need to satisfy (3). Computing, we find

$$(A - \lambda)\vec{v}_2 = \begin{bmatrix} -3 & -3 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \end{bmatrix} \neq \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \vec{v}_1.$$

But recall that we have freedom to choose the free variables, so if instead of $a = 1$ in the solution $\vec{v}_1$ we had chosen $a = -3$, then

$$\vec{v}_1 = \begin{bmatrix} -3 \\ 3 \end{bmatrix}$$

and (3) is satisfied (of course, we could instead keep $\vec{v}_1 = (1, -1)$ and choose $a = -\frac{1}{3}$, $b = 0$, for $\vec{v}_2$).

One solution is then

$$\vec{x}_1 = \vec{v}_1 e^{\lambda t} = \begin{bmatrix} -3 \\ 3 \end{bmatrix} e^{4t},$$

and another (linearly independent) one is

$$\vec{x}_2 = (t\vec{v}_1 + \vec{v}_2)e^{\lambda t} = \left(t \begin{bmatrix} -3 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) e^{4t}.$$

Let us verify that $\vec{x}_2$ is indeed a solution. Write

$$\vec{x}_2 = \left(t \begin{bmatrix} -3 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) e^{4t} = \begin{bmatrix} (-3t + 1)e^{4t} \\ 3te^{4t} \end{bmatrix}.$$

Differentiating,

$$\vec{x}_2' = \begin{bmatrix} \left((-3t + 1)e^{4t}\right)' \\ \left(3te^{4t}\right)' \end{bmatrix} = \begin{bmatrix} -3e^{4t} + 4(-3t + 1)e^{4t} \\ 3e^{4t} + 12te^{4t} \end{bmatrix} = \begin{bmatrix} 1 - 12t \\ 3 + 12t \end{bmatrix} e^{4t}.$$

On the other hand,

$$A\,\vec{x}_2 = \begin{bmatrix} 1 & -3 \\ 3 & 7 \end{bmatrix} \begin{bmatrix} (-3t + 1)e^{4t} \\ 3te^{4t} \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 3 & 7 \end{bmatrix} \begin{bmatrix} -3t + 1 \\ 3t \end{bmatrix} e^{4t}$$

$$= \begin{bmatrix} -3t+1-3\times 3t \\ 3(-3t+1)+7\times 3t \end{bmatrix} e^{4t} = \begin{bmatrix} 1-12t \\ 3+12t \end{bmatrix} e^{4t}.$$

Hence

$$\vec{x}_2' = \begin{bmatrix} 1-12t \\ 3+12t \end{bmatrix} e^{4t} = A\,\vec{x}_2 = \begin{bmatrix} 1-12t \\ 3+12t \end{bmatrix} e^{4t}$$

as desired.

URL: `http://www.disconzi.net/Teaching/MAT196-Spring-15/MAT196-Spring-15.html`