VANDERBILT UNIVERSITY
MATH 196 — EXAMPLES OF SECTION 3.6

Question 1. Find the determinant of

$$A = \begin{bmatrix} 3 & 5 & 6 \\ 2 & 4 & 3 \\ 2 & 3 & 5 \end{bmatrix}$$

Question 2. Solve the system

$$A\vec{x} = \vec{b},$$

where A is the matrix of question 1 and

$$\vec{b} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix},$$

by using the inverse of A.

Question 3. Solve the system of question 2 using Cramer's rule.

SOLUTIONS.

1. We have

$$A_{11} = \begin{bmatrix} 4 & 3 \\ 3 & 5 \end{bmatrix} \Rightarrow \det(A_{11}) = 4 \cdot 5 - 3 \cdot 3 = 11,$$

$$A_{12} = \begin{bmatrix} 2 & 3 \\ 2 & 5 \end{bmatrix} \Rightarrow \det(A_{12}) = 2 \cdot 5 - 3 \cdot 2 = 4,$$

$$A_{13} = \begin{bmatrix} 2 & 4 \\ 2 & 3 \end{bmatrix} \Rightarrow \det(A_{13}) = 2 \cdot 3 - 2 \cdot 4 = -2.$$

Hence,

$$\det(A) = 3\det(A_{11}) - 5\det(A_{12}) + 6\det(A_{13}) = 33 - 20 - 12 = 1.$$

2. Write

$$\left[\begin{array}{ccc|ccc} 3 & 5 & 6 & 1 & 0 & 0 \\ 2 & 4 & 3 & 0 & 1 & 0 \\ 2 & 3 & 5 & 0 & 0 & 1 \end{array}\right]$$

and apply Gauss-Jordan elimination.

$$\left[\begin{array}{ccc|ccc} 3 & 5 & 6 & 1 & 0 & 0 \\ 2 & 4 & 3 & 0 & 1 & 0 \\ 2 & 3 & 5 & 0 & 0 & 1 \end{array}\right] \quad \underset{\sim}{L_1 \leftarrow L_1 - L_2} \quad \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & -1 & 0 \\ 2 & 4 & 3 & 0 & 1 & 0 \\ 2 & 3 & 5 & 0 & 0 & 1 \end{array}\right] \quad \underset{\sim}{L_2 \leftarrow L_2 - L_3}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & -1 & 0 \\ 0 & 1 & -2 & 0 & 1 & -1 \\ 2 & 3 & 5 & 0 & 0 & 1 \end{array}\right] \underset{\sim}{L_3 \leftarrow L_3 - 2L_1} \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & -1 & 0 \\ 0 & 1 & -2 & 0 & 1 & -1 \\ 0 & 1 & -1 & -2 & 2 & 1 \end{array}\right]$$

$$\underset{\sim}{L_3 \leftarrow L_3 - L_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & -1 & 0 \\ 0 & 1 & -2 & 0 & 1 & -1 \\ 0 & 0 & 1 & -2 & 1 & 2 \end{array}\right] \underset{\sim}{\begin{array}{c} L_2 \leftarrow 2L_3 + L_2 \\ L_2 \leftarrow -3L_3 + L_1 \end{array}}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 7 & -4 & -6 \\ 0 & 1 & 0 & -4 & 3 & 3 \\ 0 & 0 & 1 & -2 & 1 & 2 \end{array}\right] \underset{\sim}{L_1 \leftarrow L_1 - L_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 11 & -7 & -9 \\ 0 & 1 & 0 & -4 & 3 & 3 \\ 0 & 0 & 1 & -2 & 1 & 2 \end{array}\right]$$

So

$$A^{-1} = \begin{bmatrix} 11 & -7 & -9 \\ -4 & 3 & 3 \\ -2 & 1 & 2 \end{bmatrix}.$$

Therefore

$$\vec{x} = A^{-1}\vec{b} = \begin{bmatrix} 11 & -7 & -9 \\ -4 & 3 & 3 \\ -2 & 1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -14 \\ 6 \\ 2 \end{bmatrix}.$$

3. Replace the first column of A with $\vec{b}$ to get:

$$A_1(\vec{b}) = \begin{bmatrix} 0 & 5 & 6 \\ 2 & 4 & 3 \\ 0 & 3 & 5 \end{bmatrix}.$$

Analogously, replacing the second and third column produces

$$A_2(\vec{b}) = \begin{bmatrix} 3 & 0 & 6 \\ 2 & 2 & 3 \\ 2 & 0 & 5 \end{bmatrix},$$

and

$$A_3(\vec{b}) = \begin{bmatrix} 3 & 5 & 0 \\ 2 & 4 & 2 \\ 2 & 3 & 0 \end{bmatrix}$$

Then

$$\det(A_1(\vec{b})) = 0 \cdot \det \begin{bmatrix} 4 & 2 \\ 3 & 0 \end{bmatrix} - 2 \cdot \det \begin{bmatrix} 5 & 6 \\ 3 & 5 \end{bmatrix} + 0 \cdot \det \begin{bmatrix} 5 & 6 \\ 4 & 3 \end{bmatrix} = -14.$$

Analogously,

$$\det(A_2(\vec{b})) = -0 \cdot \det \begin{bmatrix} 2 & 3 \\ 2 & 50 \end{bmatrix} + 2 \cdot \det \begin{bmatrix} 3 & 6 \\ 2 & 5 \end{bmatrix} - 0 \cdot \det \begin{bmatrix} 3 & 6 \\ 2 & 3 \end{bmatrix} = 6,$$

$$\det(A_3(\vec{b})) = 0 \cdot \det \begin{bmatrix} 2 & 4 \\ 2 & 3 \end{bmatrix} - 2 \cdot \det \begin{bmatrix} 3 & 5 \\ 2 & 3 \end{bmatrix} + 0 \cdot \det \begin{bmatrix} 3 & 5 \\ 2 & 4 \end{bmatrix} = 2.$$

We finally obtain

$$x_1 = \frac{\det(A_1(\vec{b}))}{\det(A)} = \frac{-14}{1} = -14,$$
$$x_2 = \frac{\det(A_2(\vec{b}))}{\det(A)} = \frac{6}{1} = 6,$$
$$x_3 = \frac{\det(A_3(\vec{b}))}{\det(A)} = \frac{2}{1} = 2,$$

in accordance to what we found in question 2.

URL: `http://www.disconzi.net/Teaching/MAT196-Spring-15/MAT196-Spring-15.html`