VANDERBILT UNIVERSITY
MATH 196 — DIFFERENTIAL EQUATIONS WITH LINEAR ALGEBRA
SOLUTIONS TO THE PRACTICE MIDTERM.

Question 1. Classify the differential equations below as linear or non-linear and state their order.
(a) $y' + y^2 = 0$
(b) $\frac{d^2x}{dt^2} + 25x = \cos(t)$
(c) $yy'' = \sqrt{y}$
(d) $e^{\sin x^2} \frac{dy}{dx} + xy = e^{-x}$
(e) $e^{\cos x^4} \frac{dy}{dx} y = e^{-x}$

Solution.
(a) Non-linear first order.
(b) Linear second order.
(c) Non-linear second order.
(d) Linear first-order.
(e) Non-linear first order.

Question 2. The acceleration of an object moving in a straight line is proportional to the logarithm of the time elapsed since its departure. Find an equation for its position after time t. Is this a well defined problem?

Solution. Write

$$\frac{dv}{dt} = k \ln t \Rightarrow \int dv = k \int \ln t \, dt,$$

so

$$v(t) = kt \ln t - kt + v_0.$$

Notice that this is defined at $t = 0$ since $t \ln t \to 0$ as $t \to 0^+$. Integrating again gives

$$x(t) = -\frac{3}{4}kt^2 + \frac{1}{2}kt^2 \ln t + v_0 t + x_0,$$

where again we see that this is well-defined at $t = 0$.

Question 3. A $300\,\ell$ tank initially contains $10\,kg$ of salt dissolved in $100\,\ell$ of water. Brine containing $2\,kg/\ell$ of salt flows into the tank at the rate $4\,\ell/\min$, and the well-stirred mixture flows out of the tank at the rate $2\,\ell/\min$. How much salt does the tank contain when 80% of its capacity is full?

Solution. Let $x(t)$ and $V(t)$ be respectively the amount of salt in the tank and the volume at time t. Then

$$\frac{dx}{dt} = in - out = 2\,kg/\ell \times 4\,\ell/\min - \frac{x(t)\,kg}{V(t)\,\ell} \times 2\,\ell/\min .$$

The volume at time t is $V(t) = 100 + 4t - 2t = 100 + 2t$, so

$$\frac{dx}{dt} + \frac{x}{50 + t} = 8.$$

This is a linear first order equation, where the initial condition is $x(0) = 10$. Using the formula derived in class (also in page 49 of the textbook) we find

$$x(t) = \frac{4(t^2 + 100t + 125)}{50 + t}.$$

The tank will be 80 % full when $V(t) = 100 + 2t = 240$, so $t = 70$. Then $x(70) = 601.25\,kg$.

Question 4. Solve the following differential equations:
(a) $y' = -\frac{2xy^3+e^x}{3x^2y^2+\sin y}$
(b) $-x^2y' + xy^2 + 3y^2 = 0$
(c) $x^2y' = xy + y^2$
(d) $x^3 + 3y - xy' = 0.$
(e) $y' = x^2 - 2xy + y^2$

Solution.

(a). This is an exact equation. Using the methods of section 1.6 we find the (implicit) solution $x^2y^3 + e^x - \cos y = C$.

(b) This is a separable equation. Separating variables and integrating we find $y = \frac{x}{3-x\ln x-Cx}$.

(c) This is a homogeneous equation. Using the methods of section 1.6 we find the solution $y = \frac{x}{C-\ln x}$.

(d) Writing the equation as $y' - \frac{3}{x}y = x^3$ we obtain that this is a linear equation. Using the formula for linear equations derived in class (also in page 49 of the textbook) we find $y = x^3(\ln x + C)$.

(e) Making the substitution $v = y - x$ we obtain $v' = v^2$, which is a separable equation for v. Solving and returning to y gives the (implicit) solution $y - x - 1 = Ce^{2x}(y - x + 1)$.

Question 5. Consider a second order homogeneous linear differential equation. Show that any linear combination of two solutions is also a solution. Can you make a similar statement for higher order equations?

Solution. Write

$$y'' + a(x)y' + b(x)y = 0.$$

Let y_1 and y_2 be two solutions, and c_1 and c_2 two constants. Set $w = c_1y_1 + c_2y_2$. Then

$$\begin{aligned} w'' + a(x)w' + b(x)w &= (c_1y_1 + c_2y_2)'' + a(x)(c_1y_1 + c_2y_2)' + b(x)(c_1y_1 + c_2y_2) \\ &= c_1(y_1'' + a(x)y_1' + b(x)y_1) + c_2(y_2'' + a(x)y_2' + b(x)y_2) = 0 + 0 \\ &= 0, \end{aligned}$$

hence w is also a solution.

Question 6. Solve the linear systems below, when possible.

(a)

$$\begin{cases} 3x + 5y - z = 13 \\ 2x + 7y + z = 28 \\ x + 7y + 2z = 32 \end{cases}$$

Solution. The augmented matrix of the system is

$$\left[\begin{array}{ccc:c} 3 & 5 & -1 & 13 \\ 2 & 7 & 1 & 18 \\ 1 & 7 & 2 & 32 \end{array}\right]$$

Applying Gauss-Jordan elimination we find

$$\left[\begin{array}{ccc:c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

So that $x = 1$, $y = 3$, $z = 5$.

(b)

$$\begin{cases} 2x + 3y + 7z = 15 \\ x + 4y + z = 20 \\ x + 2y + 3z = 10 \end{cases}$$

Solution. The augmented matrix of the system is

$$\left[\begin{array}{ccc:c} 2 & 3 & 7 & 15 \\ 1 & 4 & 1 & 20 \\ 1 & 2 & 3 & 10 \end{array}\right]$$

Applying Gauss-Jordan elimination we find

$$\left[\begin{array}{ccc:c} 1 & 0 & 5 & 0 \\ 0 & 1 & -1 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore z is a free variable, and solutions are given by $x = -5z$, $y = 5 + z$, and z can be any real number.

(c)

$$\begin{cases} x - 3y + 2z = 6 \\ x + 4y - z = 4 \\ 5x + 6y + z = 20 \end{cases}$$

Solution. The augmented matrix of the system is

$$\left[\begin{array}{ccc:c} 1 & -3 & 2 & 6 \\ 1 & 4 & -1 & 4 \\ 5 & 6 & 1 & 20 \end{array}\right]$$

Applying Gauss-Jordan elimination we find

$$\left[\begin{array}{ccc:c} 1 & -3 & 2 & 6 \\ 0 & 7 & -3 & -2 \\ 0 & 0 & 0 & -\frac{4}{3} \end{array}\right]$$

The last row means $0 = -\frac{4}{3}$, hence the system is inconsistent, i.e., it has no solution.

Question 7. Let

$$A = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$$

and

$$B = \begin{bmatrix} -1 & 0 & 4 \\ 3 & -2 & 5 \end{bmatrix}.$$

Calculate whichever of the matrices AB and BA is defined.

Solution. AB is well defined, but BA is not. Computing

$$A = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix},$$

$$A = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ -2 \end{bmatrix} = \begin{bmatrix} -2 \\ -6 \end{bmatrix},$$

and

$$A = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 13 \\ 31 \end{bmatrix},$$

we find

$$AB = \begin{bmatrix} 1 & -2 & 13 \\ 5 & -6 & 31 \end{bmatrix}.$$

Question 8. Let

$$A = \begin{bmatrix} 2 & 0 & 0 & -3 \\ 0 & 1 & 11 & 12 \\ 0 & 0 & 5 & 13 \\ -4 & 0 & 0 & 7 \end{bmatrix}$$

Compute $\det A$. What can you say about A^{-1}?

Solution.

$$\begin{aligned} \det A &= 2 \det \begin{bmatrix} 1 & 11 & 12 \\ 0 & 5 & 13 \\ 0 & 0 & 7 \end{bmatrix} - (-4) \det \begin{bmatrix} 0 & 0 & -3 \\ 1 & 11 & 12 \\ 0 & 5 & 13 \end{bmatrix} \\ &= 2 \cdot 1 \cdot 5 \cdot 7 + 4(-1) \det \begin{bmatrix} 0 & -3 \\ 5 & 13 \end{bmatrix} \\ &= 70 - 4 \cdot (0 - (-15)) = 10. \end{aligned}$$

Since $\det A \neq 0$, A^{-1} exists.

Question 9. Show that the vectors $\vec{v}_1 = (2, -1, 4)$, $\vec{v}_2 = (3, 0, 1)$, and $\vec{v}_3 = (1, 2, -1)$, are linearly independent and that $\text{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \mathbb{R}^3$.

Solution. Write

$$A = [\,\vec{v}_1\,\vec{v}_2\,\vec{v}_3\,].$$

Applying Gauss-Jordan elimination we find

$$\text{rref}(A) = I,$$

therefore A is invertible. Hence $A\vec{x} = \vec{b}$ always has a solution — so $\text{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \mathbb{R}^3$ — and the solution is unique — so

$$A\vec{x} = x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 = \vec{0}$$

has only the trivial solution, and we conclude that the vectors are linearly independent.

Question 10. True or false? Justify your answer.

(a) If the system $A\vec{x} = \vec{b}$ always has a solution for any vector $\vec{b}$, then the matrix A is invertible.

(b) The set of all 3×3 invertible matrices is a subspace of the vector space of all 3×3 matrices.

(c) If $\text{rref}(A) = I$ then $\det A \neq 0$.

(d) If A is $n \times m$, and the rank of A is less than n, then there exists at least one vector $\vec{b} \in \mathbb{R}^n$ such that the system $A\vec{x} = \vec{b}$ has no solution.

(e) Let A be a $n \times m$ matrix and $\vec{b} \in \mathbb{R}^n$. The set of all vectors $\vec{x} \in \mathbb{R}^m$ that solve the system $A\vec{x} = \vec{b}$ is a subspace of $\mathbb{R}^m$ if, and only if, $\vec{b} = \vec{0}$. In particular, if $\vec{b} \neq \vec{0}$, then set of all vectors $\vec{x} \in \mathbb{R}^m$ that solve the system $A\vec{x} = \vec{b}$ is never a subspace of $\mathbb{R}^m$.

Solution.

(a) False, since A does not have to be a square matrix (if A is square though then the statement is true).

(b) False. The matrices

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

are invertible, but

$$A + B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

is not.

(c) True. If $\text{rref}(A) = I$ then A is invertible, hence $\det A \neq 0$.

(d) True. If the rank of A is less than n, then $\text{rref}(A)$ has at least one row, say, the k^{th} row, with only zero entries. Therefore the rref of the augmented matrix of a system with non-zero k^{th} entry on the last column yields an inconsistent system.

(e) True. $A\vec{x} = \vec{b}$, with $\vec{b} \neq \vec{0}$, is not a subspace because it does not contain $\vec{x} = \vec{0}$.