VANDERBILT UNIVERSITY
MATH 196 — DIFFERENTIAL EQUATIONS WITH LINEAR ALGEBRA
SOLUTIONS TO THE PRACTICE FINAL.

Remark: The formulas for the Laplace transform of commonly used functions, along with its properties, will be given in the exam. Therefore, you do not have to memorize them.

Question 1. Solve the linear systems below, when possible.

(a)

$$\begin{cases} 3x + 3y + 2z = 5 \\ 2x + 5y + 2z = 3 \\ 2x + 7y + 7z = 22 \end{cases}$$

Solution. The augmented matrix is

$$\left[\begin{array}{ccc|c} 3 & 3 & 2 & 5 \\ 2 & 5 & 2 & 3 \\ 2 & 7 & 7 & 22 \end{array}\right].$$

Applying Gauss-Jordan we find

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 5 \end{array}\right],$$

hence $x = 4$, $y = -3$, $z = 5$.

(b)

$$\begin{cases} 2x + 2y + 4z = 2 \\ x - y - 4z = 3 \\ 2 + 7y + 19z = -3 \end{cases}$$

Solution. The augmented matrix is

$$\left[\begin{array}{ccc|c} 2 & 2 & 4 & 2 \\ 1 & -1 & -4 & 3 \\ 2 & 7 & 19 & -3 \end{array}\right].$$

Applying Gauss-Jordan we find

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

hence $x = 2 + z$, $y = -1 - 3z$, and z is a free variable.

(c)

$$\begin{cases} x_1 - 2x_2 - 5x_3 - 12x_4 + x_5 = 0 \\ 2x_1 + 3x_2 + 18x_3 + 11x_4 + 9x_5 = 0 \\ 2x_1 + 5x_2 + 26x_3 + 21x_4 + 11x_5 = 0 \end{cases}$$

Solution. The augmented matrix is

$$\left[\begin{array}{ccccc:c} 1 & -2 & -5 & -12 & 1 & 0 \\ 2 & 3 & 18 & 11 & 9 & 0 \\ 2 & 5 & 26 & 21 & 11 & 0 \end{array}\right].$$

Applying Gauss-Jordan we find

$$\left[\begin{array}{ccccc:c} 1 & 0 & 3 & -2 & 3 & 0 \\ 0 & 1 & 4 & 5 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right],$$

hence $x_1 = -3x_3 + 2x_4 - 3x_5$, $x_2 = -4x_3 - 5x_4 - x_5$, with x_3, x_4 and x_5 free variables.

Remark. For problem 2, notice that the rref of the given matrices was found in problem 1: we need to simply ignore the last column of the augmented matrix.

Question 2. Consider the matrix

$$A = \begin{bmatrix} 3 & 3 & 2 \\ 2 & 5 & 2 \\ 2 & 7 & 7 \end{bmatrix}.$$

Using the results and calculation of question 1,

(a) Determine whether or not $\det A = 0$.

Solution. As $rref(A) = I$, A^{-1} exists, hence $\det(A) \neq 0$.

(b) Find basis for $\text{Col}(A)$, $\text{Row}(A)$ and $\text{Ker}(A)$, if possible.

Solution. Since A is invertible, $\text{Col}(A) = \mathbb{R}^3$, and we can pick any basis of $\mathbb{R}^3$. From $rref(A) = I$ we also have $\text{Row}(A) = \mathbb{R}^3$ and $\text{Ker}(A) = \{0\}$.

(c) Determine what properties a vector $\vec{b} \in \mathbb{R}^3$ must have so that the system $A\vec{x} = \vec{b}$ always has a solution.

Solution. The system always has a unique solution since A is invertible.

(d) Again using question 1, repeat (a)-(c) with the matrix

$$B = \begin{bmatrix} 2 & 2 & 4 \\ 1 & -1 & -4 \\ 2 & 7 & 19 \end{bmatrix}.$$

Solution. The rref has only two pivot columns, therefore A is not invertible; then $\det(B) = 0$. From the rref we read off $(1, 0, -1)$ and $(0, 1, 3)$ as basis of the row space. Since the first two columns of the rref are pivot columns, the two original columns of B form a basis for the space of columns.

Next, setting equal to zero all entries on the last column of the augmented matrix , we obtain that $(1,-3,1)$ is a basis for the kernel of B. Finally, $B\vec{x}=\vec{b}$ has a solution if and only if $\vec{b}\in \text{Col}(B)$.

(e) Once again, with the help of question 1, repeat (b)-(c) with the matrix

$$C=\begin{bmatrix} 1 & -2 & -5 & -12 & 1 \\ 2 & 3 & 18 & 11 & 9 \\ 2 & 5 & 26 & 21 & 11 \end{bmatrix}.$$

Solution. Similar to (b).

Question 3. Diagonalize the matrices below, when possible.

(a)

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

Solution. The characteristic polynomial is $(\lambda-1)^2(\lambda-2)^2=0$, so the eigenvalues are $\lambda=1$ and $\lambda=2$, both with multiplicity two. The eigenvectors are $(1,0,0,0)$ and $(1,1,1,0)$ associated with $\lambda=1$ and $\lambda=2$, respectively. Since there are only two linearly independent eigenvectors for this four by four matrix, it is not diagonalizable.

(b)

$$\begin{bmatrix} 2 & 0 & 0 \\ -6 & 11 & 2 \\ 6 & -15 & 0 \end{bmatrix}.$$

Solution. The characteristic polynomial is $-\lambda^3+13\lambda^2-52\lambda+60=-(\lambda-2)(\lambda-5)(\lambda-6)=0$. Notice that since all eigenvalues are distinct, we already know at this point that the matrix is diagonalizable. The eigenvectors associated with $\lambda=2$, $\lambda=5$ and $\lambda=6$ are, respectively, $(1,0,3)$, $(0,-1,3)$, $(0,-2,5)$. The matrices S and D are

$$S=\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & -2 \\ 3 & 3 & 5 \end{bmatrix},\quad D=\begin{bmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 6 \end{bmatrix}.$$

(c)

$$\begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

Solution. The characteristic polynomial is $(\lambda+1)^2(\lambda-1)^2=0$. Therefore $\lambda=-1$ and $\lambda=1$ are eigenvalues with multiplicity two. The corresponding eigenvectors are $(0,0,0,1)$, $(1,1,1,0)$, $(0,1,0,0)$ and $(1,0,0,0)$. Since there are four linearly independent eigenvectors, the matrix is diagonalizable and

$$S=\begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix},\quad D=\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Question 4. For each diagonalizable matrix of the question 3, compute its determinant using the properties of the determinant and your answer to that question.

Solution. If a matrix A is diagonalizable, then $A = SDS^{-1}$, where D is a diagonal matrix with the eigenvalues along the diagonal. Taking the determinant and using its properties

$$\det(A) = \det(SDS^{-1}) = \det S \det D \det(S^{-1}) = \det S \det D \frac{1}{\det S} = \det D.$$

But $\det D$ is simply the product of the eigenvalues. Hence we obtain 6 and 1 for the diagonalizable matrices of the previous problem.

Question 5. Prove or give a counter-example: every invertible matrix is diagonalizable.

Solution. The matrix of problem 3a is invertible (its determiant is easily seen to be 4 since the matrix is upper triangular), but not diagonalizable.

Question 6. Recall that two matrices A and B are said to be similar if there exists an invertible matrix matrix S such that $A = S^{-1}BS$. Suppose A and B are two diagonalizable matrices with the same eigenvalues (with the same multiplicities). Show that A and B are similar. *Hint:* A and B are similar to diagonal matrices D_1 and D_2, respectively, since they are diagonalizable by hypothesis. Can you see what the relation between D_1 and D_2 is?

Solution. By hypothesis

$$D_1 = S_1^{-1}AS_1$$

and

$$D_2 = S_2^{-1}BS_2.$$

But $D_1 = D_2$, also by hypothesis, so

$$S_1^{-1}AS_1 = S_2^{-1}BS_2 \Rightarrow A = S_1S_2^{-1}BS_2S_1^{-1} = \left(S_2S_1^{-1}\right)^{-1}BS_2S_1^{-1}.$$

Hence

$$A = S^{-1}BS.$$

with $S = S_2S_1^{-1}$.

Question 7. Find the general solution of the differential equations below.

(a)

$$y' = 1 + x^2 + y^2 + x^2y^2.$$

Solution. This is a separable equation. We find $y = \tan(C + x + \frac{x^3}{3})$.

(b)

$$2x^2y - x^3y' = y^3.$$

Solution. This is a homogeneous equation to which we can apply the substitution $v = \frac{y}{x}$. The solution is $y^2 = Cx^2(x^2 - y^2)$.

(c)

$$y'' - 6y' + 13y = xe^{3x}\sin(2x).$$

Solution.

$$y = e^{3x}\big(c_1\cos(2x) + c_2\sin(2x)\big) + x\Big((Ax+B)e^{3x}\cos(2x) + (Cx+D)e^{3x}\sin(2x)\Big).$$

(d)

$$y'''' - 2y'' + y = x^2\cos x + \pi.$$

Solution.

$$y = (c_1 + c_2x)e^{-x} + (c_x + c_4x)e^{x} + (Ax^2 + Bx + C)\cos x + (Dx^2 + Ex + F)\sin x + G.$$

(e)

$$y'' + 4y = \sin^2 x.$$

Solution.

$$y = c_1\cos(2x) + c_2\sin(2x) + \frac{1 - x\sin(2x)}{8}.$$

Question 8. Solve the system

$$\vec{x}' = A\vec{x}$$

for each one of the matrices in question 3.

Solution a. This is a case with a defect. Using the recipe from section 7.5, we find two other linearly independent vectors $(0, 1, 0, 0)$ and $(0, 0, 0, 1)$ associated with the eigenvalues 1 and 2, respectively. We can now proceed as usual to write the solutions

$$\vec{x}_1 = \begin{bmatrix}1\\0\\0\\0\end{bmatrix} e^t,\ \vec{x}_2 = \begin{bmatrix}1\\0\\0\\0\end{bmatrix} te^t + \begin{bmatrix}0\\1\\0\\0\end{bmatrix} e^t,$$

$$\vec{x}_3 = \begin{bmatrix}1\\1\\1\\0\end{bmatrix} e^{2t},\ \vec{x}_4 = \begin{bmatrix}1\\1\\1\\0\end{bmatrix} te^{2t} + \begin{bmatrix}0\\0\\0\\1\end{bmatrix} e^{2t}.$$

Solution b. Three linearly independent solutions are

$$\vec{x}_1 = \begin{bmatrix}1\\0\\3\end{bmatrix} e^{2x},\ \vec{x}_2 = \begin{bmatrix}0\\-1\\3\end{bmatrix} e^{5x},\ \vec{x}_1 = \begin{bmatrix}0\\-2\\5\end{bmatrix} e^{6x}.$$

Solution c. Four linearly independent solutions are

$$\vec{x}_1 = \begin{bmatrix}0\\0\\0\\1\end{bmatrix} e^{-x},\ \vec{x}_2 = \begin{bmatrix}1\\1\\1\\0\end{bmatrix} e^{-x},\ \vec{x}_3 = \begin{bmatrix}0\\1\\0\\0\end{bmatrix} e^{x},\ \vec{x}_4 = \begin{bmatrix}1\\0\\0\\0\end{bmatrix} e^{x}.$$

Question 9. A mass of $5\,kg$ stretches a spring $10\,cm$. The mass is acted on by an external force of $10\sin(t/2)\,N$ and moves in a medium that imparts a viscous force of $2\,N$ when the speed of the mass is $4\,cm/s$. If the mass is set in motion from its equilibrium position with an initial velocity of $3\,cm/s$, determine its position x as a function of the time t.

Solution. The equation is

$$5x'' + 50x' + 490x = 10\sin\frac{t}{2},$$

or

$$x'' + 10x' + 98x = 2\sin\frac{t}{2}.$$

The characteristic roots are

$$\lambda = -5 \pm \sqrt{73}i.$$

Hence $e^{-5t}\cos(\sqrt{73}t)$ and $e^{-5t}\sin(\sqrt{73}t)$ are two linearly independent solutions. The particular solution is

$$y_p = A\cos\frac{t}{2} + B\sin\frac{t}{2}.$$

We find after some algebra, $A = -40/153\,281$, $B = 782/153\,281$.

The initial conditions are $x(0) = 0$ and $x'(0) = 0.03$, which can then be used to find the constants c_1 and c_2 in

$$x(t) = c_1 e^{-5t}\cos(\sqrt{73}t) + c_2 e^{-5t}\sin(\sqrt{73}t) + A\cos\frac{t}{2} + B\sin\frac{t}{2}.$$

Question 10. Consider two block of masses m_1 and m_2, respectively. The first block is attached to a spring of constant k_1 which, in turn, is attached to a wall, while the second block is connected to the first one by a second spring whose constant equals k_2. Let x_1 and x_2 denote the position of blocks one and two, respectively, as measured with respect to the wall. The situation is as illustrated in the picture below. Write an initial value problem which determines the motion of the system (disregard friction).

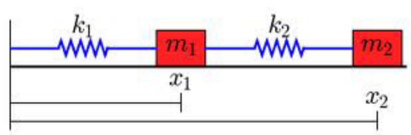

FIGURE 1. Mass-spring system of question 10.

Solution. It suffices to write a system for the displacements y_1 and y_2 with respect to the equilibrium positions of the blocks one and two, since if these, when measured with respect to the wall, are given, respectively, by ℓ_1 and ℓ_2, then $x_1 = \ell_1 + y_1$ and $x_2 = \ell_2 + y_2$. Proceeding as we did in class, we easily find

$$m_1 y_1'' = -k_1 y_1 - k_2 y_1 + k_2 y_2$$
$$m_2 y_2'' = -k_2 y_2 + k_2 y_1,$$
$$y_1(0) = a, y_1'(0) = b, y_2(0) = c, y_2'(0) = d.$$

Question 11. Repeat the previous problem for the following system:

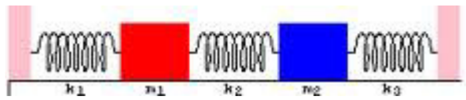

FIGURE 2. Mass-spring system of question 11.

Solution. Similar to the previous problem.

$$m_1 y_1'' = -k_1 y_1 - k_2 y_1 + k_2 y_2$$
$$m_2 y_2'' = k_2 y_1 - k_2 y_2 - k_3 y_2$$
$$y_1(0) = a, y_1'(0) = b, y_2(0) = c, y_2'(0) = d.$$

Question 12. Use Laplace transforms to solve the initial value problems below.
(a)

$$\begin{cases} x'' - 6x' + 8x = 2, \\ x(0) = x'(0) = 0. \end{cases}$$

Solution.

$$X(s) = \frac{2}{s(s^2 - 6s + 8)} = \frac{1}{4s} + \frac{1}{4}\frac{1}{s-4} - \frac{1}{2}\frac{1}{s-2},$$

$$x(t) = \frac{1}{4}(1 + e^{4t} - 2e^{2t}).$$

(b)

$$\begin{cases} x'''' + 13x'' + 36x = 0, \\ x(0) = x''(0) = 0, x'(0) = 2, x'''(0) = -13. \end{cases}$$

Solution.

$$X(s) = \frac{2s^2 + 13}{s^4 + 13s^2 + 36} = \frac{1}{s^2 + 4} + \frac{1}{s^2 + 9},$$

$$x(t) = \frac{1}{2}\sin(2t) + \frac{1}{3}\sin(3t).$$

(c)

$$\begin{cases} x'' + 6x' + 18x = \cos(2t), \\ x(0) = 1, x'(0) = -1. \end{cases}$$

Solution.

$$X(s) = \frac{s+5}{s^2 + 6s + 18} + \frac{s}{(s^2+4)(s^2 + 6s + 18)} = \frac{1}{170}\left(\frac{7s + 12}{s^2 + 4} + \frac{163(s+3)}{(s+3)^2 + 9} + \frac{307}{(s+3)^2 + 9}\right),$$

$$x(t) = \frac{1}{170}\Big(7\cos(2t) + 6\sin(2t)\Big) + \frac{1}{510}e^{-3t}\Big(489\cos(3t) + 307\sin(3t)\Big).$$

(d)

$$\begin{cases} x'' + x' + y' + 2x - y = 0, \\ y'' + x' + y' + 4x - 2y = 0, \\ x(0) = y(0) = 1, x'(0) = y'(0) = 0. \end{cases}$$

Solution.

$$X(s) = \frac{s^2 + 3s + 2}{s^3 + 3s^2 + 3s}$$
$$= \frac{2}{3s} + \frac{1}{3}\frac{s + 3/2}{(s + 3/2)^2 + (\sqrt{3}/2)^2} + \frac{\sqrt{3}}{3}\frac{\sqrt{3}/2}{(s + 3/2)^2 + (\sqrt{3}/2)^2}$$

$$Y(s) = \frac{-s^3 - 2s^2 + 2s + 4}{s^3 + 3s^2 + 3s}$$
$$= \frac{28}{21s} - \frac{9}{21}\frac{1}{s-1} + \frac{2}{21}\frac{s + 3/2}{(s + 3/2)^2 + (\sqrt{3}/2)^2} + \frac{8\sqrt{3}}{21}\frac{\sqrt{3}/2}{(s + 3/2)^2 + (\sqrt{3}/2)^2}.$$

$$x(t) = \frac{1}{3}\Big(2 + e^{-3t/2}\cos(\sqrt{3}t/2) + e^{-3t/2}\sqrt{3}\sin(\sqrt{3}t/2)\Big),$$
$$y(t) = \frac{1}{21}\Big(28 - 9e^t + 2e^{-3t/2}\cos(\sqrt{3}t/2) + 8\sqrt{3}e^{-3t/2}\sin(\sqrt{3}t/2)\Big).$$