VANDERBILT UNIVERSITY
MATH 196 — DIFFERENTIAL EQUATIONS WITH LINEAR ALGEBRA
PRACTICE MIDTERM II.

Question 1. Determine whether or not the given vectors form a basis of $\mathbb{R}^n$.
(a) $v_1 = (3, -1, 2)$, $v_2 = (6, -2, 4)$, $v_3 = (5, 3, -1)$.
(b) $v_1 = (3, -7, 5, 2)$, $v_2 = (1, -1, 3, 4)$, $v_3 = (7, 11, 3, 13)$.
(c) $v_3 = (1, 0, 0, 0)$, $v_2 = (0, 3, 0, 0)$, $v_3 = (0, 0, 7, 6)$, $v_4 = (0, 0, 4, 5)$.

Solution. a) Compute

$$\det[v_1\, v_2\, v_3] = \begin{bmatrix} 3 & 5 & 5 \\ -1 & -2 & 3 \\ 2 & 4 & -1 \end{bmatrix} = 0.$$

The vectors are linearly dependent, hence do not form a basis.
b) A basis of $\mathbb{R}^4$ must contain four vectors, so this is not a basis.
c) Compute

$$\det[v_1\, v_2\, v_3\, v_4] = 66,$$

hence we have four linearly independent vectors in $\mathbb{R}^4$ and therefore a basis.

Question 2. Consider the set W of all vectors $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ such that $x_1 = x_2 + x_3 + x_4$. Is W a sub-space of $\mathbb{R}^4$? In case yes, find a basis for W.

Solution. Any such vector can be written as $x_2(1, 1, 0, 0) + x_3(1, 0, 1, 0) + x_4(1, 0, 0, 1)$, which clearly forms a subspace with $(1, 1, 0, 0)$, $(1, 0, 1, 0)$ $(1, 0, 0, 1)$ as basis.

Question 3. Find a basis for the solution space of the linear system

$$\begin{cases} x_1 + 3x_2 - 4x_3 - 8x_4 + 6x_5 = 0 \\ x_1 + 2x_3 + x_4 + x_5 = 0 \\ 2x_1 + 7x_2 - 10x_3 - 19x_4 + 13x_5 = 0 \end{cases}$$

Solution. The rref of the matrix of the system is

$$\begin{bmatrix} 1 & 0 & 2 & 1 & 3 \\ 0 & 1 & -2 & -3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

A basis is then given by $(-2, 2, 1, 0, 0)$, $(-1, 3, 0, 1, 0)$, and $(-3, -1, 0, 0, 1)$.

Question 4. Let $\{v_1, v_2, \dots, v_n\}$ be a basis of $\mathbb{R}^n$, and let A be an invertible $n \times n$ matrix. Consider the vectors $u_1 = Av_1$, $u_2 = Av_2$, $\dots$, $u_n = Av_n$. Prove that $\{u_1, u_2, \dots, u_n\}$ is also a basis of $\mathbb{R}^n$.

Solution. Since the set $\{u_1, u_2, \dots, u_n\}$ contains n vectors, it suffices to show that they are linearly independent. Let $c_1, \dots, c_n$ be such that

$$c_1 u_1 + c_2 u_2 + \cdots c_n u_n = 0,$$

we have to show that all the constants are equal to zero. But

$$\begin{aligned} c_1u_1 + c_2u_2 + \cdots c_nu_n &= c_1Av_1 + c_2Av_2 + \cdots c_nAv_n \\ &= A(c_1v_1 + c_2v_2 + \cdots c_nv_n) \\ &= 0. \end{aligned}$$

Denote by $w = c_1v_1 + c_2v_2 + \cdots c_nv_n$, so we have

$$Aw = 0.$$

But because A is invertible, we conclude that $w = 0$, i.e.

$$c_1v_1 + c_2v_2 + \cdots c_nv_n = 0.$$

Invoking that $\{v_1, v_2, \ldots, v_n\}$ is a basis of $\mathbb{R}^n$, we obtain that $c_1 = c_2 = \cdots = c_n = 0$, as desired.

Question 5. Let u and v be arbitrary vectors in a vector space V. Recall that the norm or length of a vector is defined by $\| v \| = \sqrt{\langle v, v \rangle}$, where $\langle\, , \rangle$ is denotes an inner product on V. Show that
(a)

$$\| u + v \|^2 + \| u - v \|^2 = 2 \| u \|^2 + 2 \| v \|^2 .$$

(b)

$$\| u + v \|^2 - \| u - v \|^2 = 4\langle u, v \rangle.$$

Solution. Direct computation.

Question 6. Let $S = \{u_1, u_2\}$ and $T = \{v_1, v_2\}$ be linearly independent sets of vectors such that each u_i in S is orthogonal to every vector v_j in T. Shot that u_1, u_2, v_1, v_2 are linearly independent.

Solution. Consider $\{u_1, u_2, v_1\}$. We claim that this set is linearly independent. Since u_1 and u_2 are linearly independent by hypothesis, it suffices to show that v_1 does not belong to $\text{span}\{u_1, u_2\}$. If

$$c_1u_1 + c_2u_2 = v_1$$

then, taking inner product with v_1 we obtain

$$\| v_1 \|^2 = 0,$$

which implies that $v_1 = 0$ — but this cannot be the case since T is linearly independent and therefore it does not contain the zero vector. Analogously we obtain that v_2 cannot be spanned by u_1 and u_2. A similar argument finally shows that $\{u_1, u_2, v_1, v_2\}$ is linearly independent.

Question 7. Give the form of the particular solution for the given differential equations. You do not have to find the values of the constants of the particular solution.
(a) $y'' + 2y' - 3y = \cos x$.
(b) $y''' - 3y' - 2y = e^{-x} + 1$.
(c) $y'''' + 50y'' + 625y = \sin(5x)$.
(d) $y'''' + 2y''' - 3y'' - 4y' + 4y = xe^x + e^{2x} + x^3$.

Solution. a) The characteristic equation is

$$\lambda^2 + 2\lambda - 3 = (\lambda - 1)(\lambda + 3) = 0.$$

Hence $y_1 = e^x$ and $y_2 = e^{-3x}$ are linearly independent solutions. Since these do not involve $\cos x$, we have

$$y_p = A\cos x + B\sin x.$$

b) The characteristic equation is

$$\lambda^3 - 3\lambda - 2 = (\lambda^2 + 2\lambda + 1)(\lambda - 2) = (\lambda + 1)^2(\lambda - 2) = 0.$$

Hence $y_1 = e^{2x}$, $y_2 = e^{-x}$ and $y_3 = xe^{-x}$ are linearly independent solutions. The particular solution then takes the form

$$y_p = Ax^2e^{-x} + B.$$

c) The characteristic equation is

$$\lambda^4 + 50\lambda^2 + 625 = (\lambda^2 + 25)^2 = 0.$$

Hence $y_1 = \cos(5x)$, $y_2 = \sin(5x)$ $y_3 = x\cos(5x)$, $y_4 = x\sin(5x)$ are linearly independent solutions. The particular solution then takes the form

$$y_p = x^2(A\cos(5x) + B\sin(5x)).$$

d) The characteristic equation is

$$\lambda^4 + 2\lambda^3 - 3\lambda^2 - 4\lambda + 4 = (\lambda^2 - 2\lambda + 1)(\lambda^2 + 4\lambda + 4) = (\lambda - 1)^2(\lambda + 2)^2 = 0,$$

and from this y_p can be easily found.

Question 8. Verify that the given functions are two linearly independent solution of the corresponding homogeneous equation. Then find a particular solution solving the non-homogeneous problem
(a) $x^2y'' - 2y = 3x^2 - 1$, $x > 0$, $y_1 = x^2$, $y_2 = x^{-1}$.
(b) $(1 - x)y'' + xy' - y = \sin x$, $0 < x < 1$, $y_1 = e^x$, $y_2 = x$.

Solution. The verification is done by plugging in the given functions into the equation, while the particular solution is found with the formula

$$y_p(t) = -y_1(t)\int \frac{y_2(t)f(t)}{W(t)}\,dt + y_2(t)\int \frac{y_1(t)f(t)}{W(t)}\,dt,$$

The important thing to remember here is that in order to use the above formula we need the coefficient of y'' to be equal to one. So, for example, in (b) we need to write

$$y'' + \frac{x}{1-x}y' - \frac{1}{1-x}y = \frac{\sin x}{1-x}.$$

Using the formula we find

$$y_p(t) = -e^x\int \frac{xe^{-x}\sin x}{(1-x)^2}\,dx + x\int \frac{\sin x}{(1-x)^2}\,dx.$$

Question 9. Find the general solution of the systems below.
(a)

$$\vec{x}' = \begin{bmatrix} 5 & 5 & 2 \\ -6 & -6 & -5 \\ 6 & 6 & 5 \end{bmatrix}\vec{x}.$$

(b)

$$\vec{x}' = \begin{bmatrix} -50 & 20 \\ 100 & -60 \end{bmatrix} \vec{x}.$$

(c)

$$\vec{x}' = \begin{bmatrix} 2 & 1 & -1 \\ -4 & -3 & -1 \\ 4 & 4 & 2 \end{bmatrix} \vec{x}.$$

(d)

$$\vec{x}' = \begin{bmatrix} -13 & 40 & -48 \\ -8 & 23 & -24 \\ 0 & 0 & 3 \end{bmatrix} \vec{x}.$$

Solution. a) Look for solutions of the characteristic equation

$$\det \begin{bmatrix} 5-\lambda & 5 & 2 \\ -6 & -6-\lambda & -5 \\ 6 & 6 & 5-\lambda \end{bmatrix} = 0.$$

The solutions are

$$\lambda_1 = 0,\ \lambda_2 = 2 \pm 3i$$

As usual, we can pick only one of the two complext roots, so

$$\lambda_1 = 0,\ \lambda_2 = 2 + 3i.$$

The eigenvectors are

$$v_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix},$$

$$v_2 = \begin{bmatrix} 1+i \\ -2 \\ 2 \end{bmatrix}.$$

The solution corresponding to $\lambda_1 = 0$ then becomes,

$$x_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix},$$

while the two solutions obtained from $\lambda_2 = 2 + 3i$ are

$$x_2 = \begin{bmatrix} \cos(3t) - \sin(3t) \\ -2\cos(3t) \\ 2\cos(3t) \end{bmatrix} e^{2t},$$

$$x_3 = \begin{bmatrix} \cos(3t) + \sin(3t) \\ -2\sin(3t) \\ 2\sin(3t) \end{bmatrix} e^{2t}.$$

b) $\lambda_1 = -10$, $\lambda_2 = -100$, $v_1 = (1, 2)$, $v_2 = (2, -5)$, $x_1 = v_1 e^{\lambda_1 t}$, $x_2 = v_2 e^{\lambda_2 t}$.

c) $\lambda_1 = 1$, $\lambda_2 = \pm 2i$, $v_1 = (1,-1,0)$, $v_2 = (2+i,-3+i,3-i)$. $x_1 = e^t v_1$, $x_2 = (2\cos(2t) - \sin(2t), 3\cos(2t)+\sin(2t), 3\cos(2t)+\sin(2t))$, $x_3 = (\cos(2t)+2\sin(2t), \cos(2t)-3\sin(2t), 3\sin(2t) - \cos(2t))$.

d) The characteristic equation is

$$-\lambda^3 + 13\lambda^2 - 51\lambda + 63 = -(\lambda-3)^2(\lambda-7) = 0.$$

This gives $\lambda_1 = 3$ with multiplicity two and $\lambda_2 = 7$. We find that $(5,2,0)$, $(-3,0,1)$ are (linearly independent) eigenvectors associated with $\lambda = 3$ and $(2,1,0)$ is associated with $\lambda = 7$. Hence

$$x_1 = e^{3t}\begin{bmatrix} 5 \\ 2 \\ 0 \end{bmatrix}, \; x_2 = e^{3t}\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}, \; x_3 = e^{7t}\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}.$$

Question 10. Consider the two interconnected tanks shown in figure 1. Tank 1 initially contains $30gal$ of water and $25oz$ of salt, while tank 2 initially contains $20gal$ of water and $15oz$ of salt. Water containing $1oz/gal$ of salt flows into tank 1 at a rate of $1.5gal/min$. The mixture flows from tank 1 to tank 2 at a rate of $3gal/min$. Water containing $3oz/gal$ of salt also flows into tank 2 at a rate of $1gal/min$ (from the outside, see picture). The mixture drains from tank 2 at a rate of $4gal/min$, of which some flows back to tank 2 at a rate of $1.5gal/min$, while the remainder leaves the tank.
(a) Let $Q_1(t)$ and $Q_2(t)$, respectively, be the amount of salt in each tank at time t. Write down differential equations and initial conditions that model the flow process. Observe that the system of differential equations is non-homogeneous.
(b) Find the values of $Q_1(t)$ and $Q_2(t)$ for which the system is in equilibrium, i.e., does not change with time. Let Q_1^E and Q_2^E be the equilibrium values. Can you predict which tank will approach its equilibrium state more rapidly?
(c) Let $x_1(t) = Q_1(t) - Q_1^E$ and $x_2(t) = Q_2(t) - Q_2^E$. Determine an initial value problem for x_1 and x_2. Observe that the system of equations for x_1 and x_2 is homogeneous.
(d) Find $Q_1(t)$ and $Q_1(t)$.

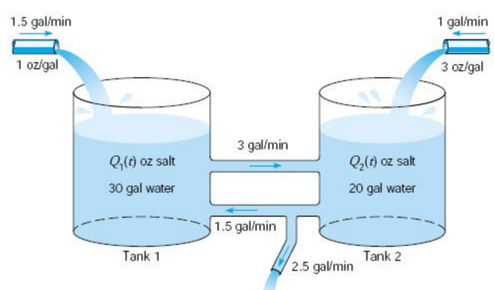

FIGURE 1. Tanks of problem 10.

Solution. The volumes of the tanks 1 and 2 are

$$V_1(t) = 30 + 1.5t - 3t + 1.5t = 30,$$
$$V_2(t) = 20 + 3t + 1t - 4t = 20.$$

$$\begin{cases} Q_1' &= 1.5 \times 1 - 3\frac{Q_1}{V_1} + 1.5\frac{Q_2}{V_2}, \\ Q_2' &= 1 \times 3 + \frac{Q_1}{V_1} - 4\frac{Q_2}{V_2}, \end{cases}$$

$$Q_1(0) = 25, Q_2(0) = 15.$$

Or

$$\begin{cases} Q_1' &= 1.5 - \frac{Q_1}{10} + \frac{1.5}{20} Q_2, \\ Q_2' &= 3 + \frac{Q_1}{20} - 4\frac{Q_2}{20}, \end{cases}$$
$$Q_1(0) = 25, Q_2(0) = 15.$$

The equilibrium is given by

$$\begin{cases} 0 &= 1.5 - \frac{Q_1}{10} + \frac{1.5}{20} Q_2, \\ 0 &= 3 + \frac{Q_1}{20} - 4\frac{Q_2}{20}, \end{cases}$$

which gives $Q_1^E = 42$, $Q_2^E = 36$.

If we write $Q = (Q_1, Q_2)$, $Q^E = (Q_1^E, Q_2^E)$ and write the system as

$$Q' = AQ + \begin{bmatrix} 1.5 \\ 3 \end{bmatrix},$$

then $x = Q - Q^E$ satisfies

$$\begin{aligned} x' = Q' &= AQ + \begin{bmatrix} 1.5 \\ 3 \end{bmatrix} \\ &= A(x + Q^E) + \begin{bmatrix} 1.5 \\ 3 \end{bmatrix} \\ &= Ax + \left(AQ^E + \begin{bmatrix} 1.5 \\ 3 \end{bmatrix} \right) \\ &= Ax + 0 = Ax, \end{aligned}$$

i.e.,

$$x' = Ax.$$