SOLUTIONS TO QUIZ 2

MATH 196.3

Problem 1. Show that $x_1 = t^2, x_2 = t^{-3}$ are solutions of

$$t^2x'' + 2tx' - 6x = 0$$

Find another solution satisfying $x(1) = 1, x'(1) = -6$.

Solution. For x_1:

$$t^2x_1'' + 2tx_1' - 6x_1 = t^2 \cdot 2 + 2t \cdot 2t - 6t^2 = (2 + 4 - 6)t^2 = 0 \quad \checkmark$$

For x_2:

$$t^2x_2'' + 2tx_2' - 6x_2 = t^2 \cdot 12t^{-5} + 2t \cdot (-3)t^{-4} - 6t^{-3} = (12 - 6 - 6)t^{-3} = 0 \quad \checkmark$$

Now to solve the initial value problem, we write our unknown $x(t) = c_1x_1 + c_2x_2$ and use the given data to find its coefficients:

$$1 = x(1) = c_1x_1(1) + c_2x_2(1) = c_1 + c_2$$
$$-6 = x'(1) = c_1x_1'(1) + c_2x_2'(1) = 2c_1 - 3c_2$$

The solution to this system is $c_1 = -\frac{3}{5}, c_2 = \frac{8}{5}$, so the final answer is

$$x(t) = -\frac{3}{5}t^2 + \frac{8}{5}t^{-3}$$

□

Problem 2. Find a general solution to

$$y^{(4)} + 3y'' - 4y = 0$$

Solution. The polynomial equation associated to this ODE is

$$\lambda^4 + 3\lambda^2 - 4 = 0$$

I saw more than one person try to solve this equation by moving the constant term to the right and then factoring:

$$\lambda^2(\lambda^2 + 3) = 4$$

DO NOT DO THIS. For one thing, it gives the wrong answer; for another thing, it completely misses the underlying reason why factoring polynomials helps find their roots, which is the "zero product" property of the real (and complex) numbers.

To solve this equation, we could do one of two things. First, we could observe that it is actually a *quadratic* equation in λ^2. Alternatively, we could notice that, because the coefficients sum to zero, $\lambda = 1$ is a root, and since the polynomial is even, that means $\lambda = -1$ is also. Either way, we get that this equation factors

$$(\lambda - 1)(\lambda + 1)(\lambda^2 + 4) = 0$$

so its roots are $\lambda = \pm 1$, $\lambda = \pm 2i$.

Hence the basic solutions to our original ODE are e^x, e^{-x}, $\cos(2x)$, $\sin(2x)$, and a general solution is a generic linear combination of these:

$$y = c_1 e^x + c^2 e^{-x} + c_3 \cos(2x) + c_4 \sin(2x)$$

Notice that this has *four* free parameters; you were guaranteed to lose points if you had fewer. □

Problem 3. Find a basis for the column space of

$$A = \begin{pmatrix} 3 & 0 & -3 & 2 & 2 \\ -1 & 1 & 4 & 0 & -1 \\ 4 & -2 & -10 & 5 & 7 \end{pmatrix}$$

Express each of the columns in terms of your basis.

Solution. Many of you remembered that, when extracting a basis out of a set of columns, the first step is row-reducing. Many of you reduced A to a matrix looking like this:

$$B = \begin{pmatrix} 1 & 0 & -1 & -3 & * \\ 0 & 1 & 3 & -3 & * \\ 0 & 0 & 0 & 11 & * \end{pmatrix}$$

This is where many people made the following mistake: they said that the leading ones were in columns 1 and 2, therefore the column basis was the original columns 1 and 2.

What is the problem with this? That the leading 11 in the last row is just as relevant as a leading 1 would be (and in fact, if we divided the last row by 11, which is a perfectly valid row operation, then we *would* get a leading 1 in the fourth column!) Another way to think about this: once we've gotten to the matrix B above, we can see that the *row rank* is equal to 3; but we know that row rank and column rank are equal, meaning we need to find three basis columns also.

Ok, so columns $\vec{a}_1, \vec{a}_2$, and $\vec{a}_4$ of A are a basis for the column space. How do we go about expressing column 3 and column 5 in terms of that basis? Well, it turns out that reducing all the way down to the *reduced* row echelon form

$$C = \begin{pmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 3 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

is the way to go here. Why? Because asking for coefficients c_1, c_2, c_4 such that $c_1\vec{a}_1 + c_2\vec{a}_2 + c_4\vec{a}_4 = \vec{a}_3$ is the same as reducing the submatrix

$$\begin{pmatrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_4 & \vec{a}_3 \end{pmatrix} \qquad \text{to} \qquad \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

which we've basically already done: the same reduction steps that got us C will work here. Similarly

$$\begin{pmatrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_4 & \vec{a}_5 \end{pmatrix} \quad \text{row reduces to} \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

Hence we have

$$\vec{a}_3 = 3\vec{a}_2 - \vec{a}_1 \qquad \vec{a}_5 = \vec{a}_4 - \vec{a}_2$$

The foregoing was the way you were taught to solve this problem in the book. However, there is a tricky way too, once you've row-reduced to B. Since the rank of B is 3, we know that the column space is a 3-dimensional subspace of $\mathbb{R}^3$. But there's only one such subspace: $\mathbb{R}^3$ itself! And we know a really useful basis of $\mathbb{R}^3$, namely $\vec{e}_1, \vec{e}_2, \vec{e}_3$. With respect to *this* basis, however, our columns are

$$\begin{aligned}\vec{a}_1 &= 3\vec{e}_1 - \vec{e}_2 + 4\vec{e}_3\\ \vec{a}_2 &= \vec{e}_2 - 2\vec{e}_3\\ &\vdots\end{aligned}$$

which is more complicated than the expression in terms of the basis of columns. □

Problem 4. Find a basis for the subspace $V \leq \mathbb{R}^4$ consisting of all vectors orthogonal to $\begin{pmatrix}1\\-2\\3\\-4\end{pmatrix}$.

Solution. Call the given vector $\vec{a}$. Saying "$\vec{x}$ is orthogonal to $\vec{a}$" is a matrix equation

$$\begin{pmatrix}1 & -2 & 3 & -4\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix} = 0$$

(check: how many rows and columns should the product on the left have?). Magically, this equation is already in reduced form: we have three free parameters $x_4 = t_1, x_3 = t_2, x_2 = t_3$ forcing $x_1 = 2t_3 - 3t_2 + 4t_1$. Hence

$$\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix} = t_1\begin{pmatrix}4\\0\\0\\1\end{pmatrix} + t_2\begin{pmatrix}-3\\0\\1\\0\end{pmatrix} + t_3\begin{pmatrix}2\\1\\0\\0\end{pmatrix}$$

and the three vectors above form a basis for V. □

Problem 5. Give three linearly independent solutions of the ODE

$$y^{(3)} - 3y' - 2y = 0$$

Use the Wronskian to show they are linearly independent.

Solution. Again, some people wanted to factor the characteristic equation

$$\lambda(\lambda^2 - 3) = 2$$

and get roots $\lambda = 0, \lambda = \pm\sqrt{3}$. DO NOT DO THIS. IT IS WRONG. Instead, observe that $\lambda = -1$ is a root, and factor the polynomial into

$$(\lambda + 1)(\lambda + 1)(\lambda - 2) = 0$$

giving basis functions e^{-x}, xe^{-x}, e^{2x}.

The Wronskian of these three functions is

$$W = \det\begin{pmatrix}e^{-x} & e^{2x} & e^{-x}\cdot x\\ -e^{-x} & 2e^{2x} & e^{-x}(1-x)\\ e^{-x} & 4e^{2x} & e^{-x}(x-2)\end{pmatrix}$$

I recommend expanding along the first column; notice that every subdeterminant term will have a factor of $e^{2x}e^{-x}$, which we can bring out front:

$$\begin{aligned} W &= e^{-x}e^{2x}e^{-x}\left(2(x-2) - 4(1-x) + 1(x-2) - 4x + 1(1-x) - 2x\right) \\ &= e^{0x}\left(0x - 9\right) \\ &= -9 \end{aligned}$$

which is never 0; hence the three functions are linearly independent on the whole real line. □