VANDERBILT UNIVERSITY
MATH 196 — DIFFERENTIAL EQUATIONS WITH LINEAR ALGEBRA
PRACTICE MIDTERM II.

Question 1. Determine whether or not the given vectors form a basis of $\mathbb{R}^n$.
(a) $v_1 = (3, -1, 2)$, $v_2 = (6, -2, 4)$, $v_3 = (5, 3, -1)$.
(b) $v_2 = (3, -7, 5, 2)$, $v_2 = (1, -1, 3, 4)$, $v_3 = (7, 11, 3, 13)$.
(c) $v_3 = (1, 0, 0, 0)$, $v_2 = (0, 3, 0, 0)$, $v_3 = (0, 0, 7, 6)$, $v_4 = (0, 0, 4, 5)$.

Question 2. Consider the set W of all vectors $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ such that $x_1 = x_2 + x_3 + x_4$. Is W a sub-space of $\mathbb{R}^4$? In case yes, find a basis for W.

Question 3. Find a basis for the solution space of the linear system

$$\begin{cases} x_1 + 3x_2 - 4x_3 - 8x_4 + 6x_5 = 0 \\ x_1 + 2x_3 + x_4 + x_5 = 0 \\ 2x_1 + 7x_2 - 10x_3 - 19x_4 + 13x_5 = 0 \end{cases}$$

Question 4. Let $\{v_1, v_2, \ldots, v_n\}$ be a basis of $\mathbb{R}^n$, and let A be an invertible $n \times n$ matrix. Consider the vectors $u_1 = Av_1$, $u_2 = Av_2$, $\ldots$, $u_n = Av_n$. Prove that $\{u_1, u_2, \ldots, u_n\}$ is also a basis of $\mathbb{R}^n$.

Question 5. Let u and v be arbitrary vectors in a vector space V. Recall that the norm or length of a vector is defined by $\| v \| = \sqrt{\langle v, v \rangle}$, where $\langle\, , \,\rangle$ is denotes an inner product on V. Show that
(a)

$$\| u + v \|^2 + \| u - v \|^2 = 2 \| u \|^2 + 2 \| v \|^2 .$$

(b)

$$\| u + v \|^2 - \| u - v \|^2 = 4\langle u, v \rangle.$$

Question 6. Let $S = \{u_1, u_2\}$ and $T = \{v_1, v_2\}$ be linearly independent sets of vectors such that each u_i in S is orthogonal to every vector v_j in T. Shot that u_1, u_2, v_1, v_2 are linearly independent.

Question 7. Give the form of the particular solution for the given differential equations. You do not have to find the values of the constants of the particular solution.
(a) $y'' + 2y' - 3y = \cos x$.
(b) $y''' - 3y' - 2y = e^{-x} + 1$.
(c) $y'''' + 50y'' + 625y = \sin(5x)$.
(d) $y'''' + 2y''' - 3y'' - 4y' + 4 = xe^x + e^{2x} + x^3$.

Question 8. Verify that the given functions are two linearly independent solution of the corresponding homogeneous equation. Then find a particular solution solving the non-homogeneous problem
(a) $x^2y'' - 2y = 3x^2 - 1$, $x > 0$, $y_1 = x^2$, $y_2 = x^{-1}$.
(b) $(1 - x)y'' + xy' - y = \sin x$, $0 < x < 1$, $y_1 = e^x$, $y_2 = x$.

Question 9. Find the general solution of the systems below.

(a)

$$\vec{x}' = \begin{bmatrix} 5 & 5 & 2 \\ -6 & -6 & -5 \\ 6 & 6 & 5 \end{bmatrix} \vec{x}.$$

(b)

$$\vec{x}' = \begin{bmatrix} -50 & 20 \\ 100 & -60 \end{bmatrix} \vec{x}.$$

(c)

$$\vec{x}' = \begin{bmatrix} 2 & 1 & -1 \\ -4 & -3 & -1 \\ 4 & 4 & 2 \end{bmatrix} \vec{x}.$$

(d)

$$\vec{x}' = \begin{bmatrix} -13 & 40 & -48 \\ -8 & 23 & -24 \\ 0 & 0 & 3 \end{bmatrix} \vec{x}.$$

Question 10. Consider the two interconnected tanks shown in figure 1. Tank 1 initially contains $30gal$ of water and $25oz$ of salt, while tank 2 initially contains $20gal$ of water and $15oz$ of salt. Water containing $1oz/gal$ of salt flows into tank 1 at a rate of $1.5gal/min$. The mixture flows from tank 1 to tank 2 at a rate of $3gal/min$. Water containing $3oz/gal$ of salt also flows into tank 2 at a rate of $1gal/min$ (from the outside, see picture). The mixture drains from tank 2 at a rate of $4gal/min$, of which some flows back to tank 2 at a rate of $1.5gal/min$, while the remainder leaves the tank.
(a) Let $Q_1(t)$ and $Q_2(t)$, respectively, be the amount of salt in each tank at time t. Write down differential equations and initial conditions that model the flow process. Observe that the system of differential equations is non-homogeneous.
(b) Find the values of $Q_1(t)$ and $Q_2(t)$ for which the system is in equilibrium, i.e., does not change with time. Let Q_1^E and Q_2^E be the equilibrium values. Can you predict which tank will approach its equilibrium state more rapidly?
(c) Let $x_1(t) = Q_1(t) - Q_1^E$ and $x_2(t) = Q_2(t) - Q_2^E$. Determine an initial value problem for x_1 and x_2. Observe that the system of equations for x_1 and x_2 is homogeneous.
(d) Find $Q_1(t)$ and $Q_1(t)$.

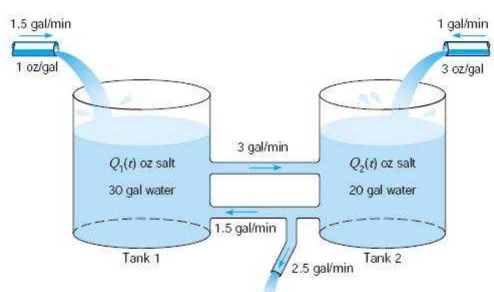

FIGURE 1. Tanks of problem 10.