VANDERBILT UNIVERSITY
MATH 196 — DIFFERENTIAL EQUATIONS WITH LINEAR ALGEBRA
PRACTICE MIDTERM.

Question 1. Classify the differential equations below as linear or non-linear and state their order.
(a) $y' + y^2 = 0$
(b) $\frac{d^2x}{dt^2} + 25x = \cos(t)$
(c) $yy'' = \sqrt{y}$
(d) $e^{\sin x^2} \frac{dy}{dx} + xy = e^{-x}$
(e) $e^{\cos x^4} \frac{dy}{dx} y = e^{-x}$

Question 2. The acceleration of an object moving in a straight line is proportional to the logarithm of the time elapsed since its departure. Find an equation for its position after time t. Is this a well defined problem?

Question 3. A $300\,\ell$ tank initially contains $10\,kg$ of salt dissolved in $100\,\ell$ of water. Brine containing $2\,kg/\ell$ of salt flows into the tank at the rate $4\,\ell/\min$, and the well-stirred mixture flows out of the tank at the rate $2\,\ell/\min$. How much salt does the tank contain when 80% of its capacity is full?

Question 4. Solve the following differential equations:
(a) $y' = -\frac{2xy^3 + e^x}{3x^2y^2 + \sin y}$
(b) $-x^2y' + xy^2 + 3y^2 = 0$
(c) $x^2y' = xy + y^2$
(d) $x^3 + 3y - xy' = 0$.
(e) $y' = x^2 - 2xy + y^2$

Question 5. Consider a second order homogeneous linear differential equation. Show that any linear combination of two solutions is also a solution. Can you make a similar statement for higher order equations?

Question 6. Solve the linear systems below, when possible.

(a)

$$\begin{cases} 3x + 5y - z = 13 \\ 2x + 7y + z = 28 \\ x + 7y + 2z = 32 \end{cases}$$

(b)

$$\begin{cases} 2x + 3y + 7z = 15 \\ x + 4y + z = 20 \\ x + 2y + 3z = 10 \end{cases}$$

(c)

$$\begin{cases} x - 3y + 2z = 6 \\ x + 4y - z = 4 \\ 5x + 6y + z = 20 \end{cases}$$

Question 7. Let

$$A = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$$

and

$$B = \begin{bmatrix} -1 & 0 & 4 \\ 3 & -2 & 5 \end{bmatrix}.$$

Calculate whichever of the matrices AB and BA is defined.

Question 8. Let

$$A = \begin{bmatrix} 2 & 0 & 0 & -3 \\ 0 & 1 & 11 & 12 \\ 0 & 0 & 5 & 13 \\ -4 & 0 & 0 & 7 \end{bmatrix}$$

Compute $\det A$. What can you say about A^{-1}?

Question 9. Show that the vectors $\vec{v}_1 = (2, -1, 4)$, $\vec{v}_2 = (3, 0, 1)$, and $\vec{v}_3 = (1, 2, -1)$, are linearly independent and that $\text{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \mathbb{R}^3$.

Question 10. True or false? Justify your answer.

(a) If the system $A\vec{x} = \vec{b}$ always has a solution for any vector $\vec{b}$, then the matrix A is invertible.

(b) The set of all 3×3 invertible matrices is a subspace of the vector space of all 3×3 matrices.

(c) If $\text{rref}(A) = I$ then $\det A \neq 0$.

(d) If A is $n \times m$, and the rank of A is less than n, then there exists at least one vector $\vec{b} \in \mathbb{R}^n$ such that the system $A\vec{x} = \vec{b}$ has no solution.

(e) Let A be a $n \times m$ matrix and $\vec{b} \in \mathbb{R}^n$. The set of all vectors $\vec{x} \in \mathbb{R}^m$ that solve the system $A\vec{x} = \vec{b}$ is a subspace of $\mathbb{R}^m$ if, and only if, $\vec{b} = \vec{0}$. In particular, if $\vec{b} \neq \vec{0}$, then set of all vectors $\vec{x} \in \mathbb{R}^m$ that solve the system $A\vec{x} = \vec{b}$ is never a subspace of $\mathbb{R}^m$.