VANDERBILT UNIVERSITY
MATH 196 — DIFFERENTIAL EQUATIONS WITH LINEAR ALGEBRA
PRACTICE FINAL.

Remark: The formulas for the Laplace transform of commonly used functions, along with its properties, will be given in the exam. Therefore, you do not have to memorize them.

Question 1. Solve the linear systems below, when possible.

(a)

$$\begin{cases} 3x + 3y + 2z = 5 \\ 2x + 5y + 2z = 3 \\ 2x + 7y + 7z = 22 \end{cases}$$

(b)

$$\begin{cases} 2x + 2y + 4z = 2 \\ x - y - 4z = 3 \\ 2 + 7y + 19z = -3 \end{cases}$$

(c)

$$\begin{cases} x_1 - 2x_2 - 5x_3 - 12x_4 + x_5 = 0 \\ 2x_1 + 3x_2 + 18x_3 + 11x_4 + 9x_5 = 0 \\ 2x_1 + 5x_2 + 26x_3 + 21x_4 + 11x_5 = 0 \end{cases}$$

Question 2. Consider the matrix

$$A = \begin{bmatrix} 3 & 3 & 2 \\ 2 & 5 & 2 \\ 2 & 7 & 7 \end{bmatrix}.$$

Using the results and calculation of question 1,

(a) Determine whether or not $\det A = 0$.

(b) Find basis for $\text{Col}(A)$, $\text{Row}(A)$ and $\text{Ker}(A)$, if possible.

(c) Determine what properties a vector $\vec{b} \in \mathbb{R}^3$ must have so that the system $A\vec{x} = \vec{b}$ always has a solution.

(d) Again using question 1, repeat (a)-(c) with the matrix

$$B = \begin{bmatrix} 2 & 2 & 4 \\ 1 & -1 & -4 \\ 2 & 7 & 19 \end{bmatrix}.$$

(e) Once again, with the help of question 1, repeat (b)-(c) with the matrix

$$C = \begin{bmatrix} 1 & -2 & -5 & -12 & 1 \\ 2 & 3 & 18 & 11 & 9 \\ 2 & 5 & 26 & 21 & 11 \end{bmatrix}.$$

Question 3. Diagonalize the matrices below, when possible.

(a)

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

(b)

$$\begin{bmatrix} 2 & 0 & 0 \\ -6 & 11 & 2 \\ 6 & -15 & 0 \end{bmatrix}.$$

(c)

$$\begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

Question 4. For each diagonalizable matrix of the question 3, compute its determinant using the properties of the determinant and your answer to that question.

Question 5. Prove or give a counter-example: every invertible matrix is diagonalizable.

Question 6. Recall that two matrices A and B are said to be similar if there exists an invertible matrix matrix S such that $A = S^{-1}BS$. Suppose A and B are two diagonalizable matrices with the same eigenvalues (with the same multiplicities). Show that A and B are similar. *Hint:* A and B are similar to diagonal matrices D_1 and D_2, respectively, since they are diagonalizable by hypothesis. Can you see what the relation between D_1 and D_2 is?

Question 7. Find the general solution of the differential equations below.

(a)

$$y' = 1 + x^2 + y^2 + x^2y^2.$$

(b)

$$2x^2y - x^3y' = y^3.$$

(c)

$$y'' - 6y' + 13y = xe^{3x}\sin(2x).$$

(d)

$$y'''' - 2y'' + y = x^2\cos x + \pi.$$

(e)

$$y'' + 4y = \sin^2 x.$$

Question 8. Solve the system

$$\vec{x}' = A\vec{x}$$

for each one of the matrices in question 3.

Question 9. A mass of $5\,kg$ stretches a spring $10\,cm$. The mass is acted on by an external force of $10\sin(t/2)\,N$ and moves in a medium that imparts a viscous force of $2\,N$ when the speed of the mass is $4\,cm/s$. If the mass is set in motion from its equilibrium position with an initial velocity of $3\,cm/s$, determine its position x as a function of the time t.

Question 10. Consider two block of masses m_1 and m_2, respectively. The first block is attached to a spring of constant k_1 which, in turn, is attached to a wall, while the second block is connected to the first one by a second spring whose constant equals k_2. Let x_1 and x_2 denote the position of blocks one and two, respectively, as measured with respect to the wall. The situation is as illustrated in the picture below. Write an initial value problem which determines the motion of the system (disregard friction).

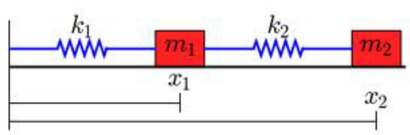

FIGURE 1. Mass-spring system of question 10.

Question 11. Repeat the previous problem for the following system:

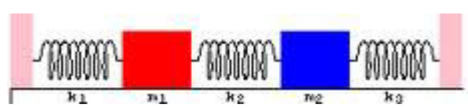

FIGURE 2. Mass-spring system of question 11.

Question 12. Use Laplace transforms to solve the initial value problems below.

(a)

$$\begin{cases} x'' - 6x' + 8x = 2, \\ x(0) = x'(0) = 0. \end{cases}$$

(b)

$$\begin{cases} x'''' + 13x'' + 36x = 0, \\ x(0) = x'{\prime}(0) = 0, x'(0) = 2, x'''(0) = -13. \end{cases}$$

(c)

$$\begin{cases} x'' + 6x' + 18x = \cos(2t), \\ x(0) = 1, x'(0) = -1. \end{cases}$$

(d)

$$\begin{cases} x'' + x' + y' + 2x - y = 0, \\ y'' + x' + y' + 4x - 2y = 0, \\ x(0) = y(0) = 1, x'(0) = y'(0) = 0. \end{cases}$$