VANDERBILT UNIVERSITY
MATH 196 — DIFFERENTIAL EQUATIONS WITH LINEAR ALGEBRA
MOTIVATION FOR VECTOR SPACES.

Source: I got this example from http://www.mtholyoke.edu/~jjlee/Teaching/notes5.pdf

Motivation. Let S be the set of all solutions to the differential equation $y'' + y = 0$. Let T be the set of all 2×3 matrices with real entries. These two sets share many common properties:

S = the set of all solutions to $y'' + y = 0$	T = the set of all 2×3 matrices
The sum of two solutions $y_1(x) = \sin x$ and $y_2(x) = \cos x$ to the differential equation, say $y_3(x) = \sin x + \cos x$, is also a solution to the equation.	$\begin{bmatrix} 1 & 2 & 3 \\ -2 & 3 & 4 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 & 2 \\ 1 & 3 & -2 \end{bmatrix}$ are in T and so is their sum $\begin{bmatrix} 1 & 2 & 5 \\ -1 & 6 & 2 \end{bmatrix}$.
The zero function is a solution to the equation.	The zero matrix $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is in T.
$y_1(x) = \sin x$ is a solution to the equation and so is any constant multiple $y_c(x) = c \sin x$. In particular $-y_1(x) = -\sin x$ is also a solution.	$\begin{bmatrix} 1 & 2 & 3 \\ -2 & 3 & 4 \end{bmatrix}$ is in T and so is $c\begin{bmatrix} 1 & 2 & 3 \\ -2 & 3 & 4 \end{bmatrix} = \begin{bmatrix} c & 2c & 3c \\ -2c & 3c & 4c \end{bmatrix}$ for every constant c. In particular $-\begin{bmatrix} 1 & 2 & 3 \\ -2 & 3 & 4 \end{bmatrix} = \begin{bmatrix} -1 & -2 & -3 \\ 2 & -3 & -4 \end{bmatrix}$ is in T.

Even though the sets S and T are totally different objects, they *resemble* each other. Due to such similarities, it is useful to study both sets S and T from the same point of view, i.e., with the same tools and techniques. What S and T have in common is that both are vector spaces, whose definition we now recall.

A vector space is a nonempty set V of elements, called vectors, together with two operations $+$ and $\cdot$, called addition and scalar multiplication, such that if $\boldsymbol{u}$, $\boldsymbol{v} \in V$, then $\boldsymbol{u} + \boldsymbol{v} \in V$, and if $\alpha \in \mathbb{R}$, $\boldsymbol{u} \in V$, then $\alpha \cdot \boldsymbol{u} \in V$. Furthermore, the following conditions are required to hold (below we write the scalar multiplication simply as $\alpha\boldsymbol{u}$ rather than $\alpha \cdot \boldsymbol{u}$ for simplicity): for all $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in V$ and $\alpha, \beta \in \mathbb{R}$,

1. $\boldsymbol{u} + \boldsymbol{v} = \boldsymbol{v} + \boldsymbol{u}$.
2. $\boldsymbol{u} + (\boldsymbol{v} + \boldsymbol{w}) = (\boldsymbol{u} + \boldsymbol{v}) + \boldsymbol{w}$.
3. There is a special element $\boldsymbol{0}$ in V such that $\boldsymbol{u} + \boldsymbol{0} = \boldsymbol{0} + \boldsymbol{u} = \boldsymbol{u}$ for all $\boldsymbol{u}$ in V.
4. $\boldsymbol{u} + (-\boldsymbol{u}) = (-\boldsymbol{u}) + \boldsymbol{u} = \boldsymbol{0}$, where $-\boldsymbol{u} = (-1)\boldsymbol{u}$.
5. $\alpha(\boldsymbol{u} + \boldsymbol{v}) = \alpha\boldsymbol{u} + \alpha\boldsymbol{v}$.
6. $(\alpha + \beta)\boldsymbol{u} = \alpha\boldsymbol{u} + \beta\boldsymbol{u}$.
7. $\alpha(\beta\boldsymbol{u}) = (\alpha\beta)\boldsymbol{u}$.
8. $1\boldsymbol{u} = \boldsymbol{u}$.