SOLUTIONS TO GRADED PROBLEMS, 1.4–1.5

MATH 196.3

Problem 1 (1.4.11)**.** Find a general solution to the ODE

$$\frac{\mathrm{d}y}{\mathrm{d}x} = xy^3$$

Solution. This equation is separable: we rewrite as

$$y^{-3}\mathrm{d}y = x\mathrm{d}x$$

Integrating gives

$$\frac{1}{-2}y^{-2} = \frac{1}{2}x^2 + C$$

or (silently replacing C)

$$y^2 = \frac{1}{C - x^2}$$

Notice, that $y \equiv 0$ is a singular solution; nonconstant solutions are given by

$$y = \pm\frac{1}{\sqrt{C - X^2}}$$

which we cannot improve on without knowing the sign of y. □

Problem 2 (1.4.54)**.** A tank shaped like a vertical cylinder begins with a depth of 9 ft of water. At noon, a plug is opened at the bottom of the tank. After one hour, the water has fallen to 4 feet. At what time will the last of the water drain from the tank?

Solution. Recall that the basic ODE that models this situation is

$$\frac{\mathrm{d}V}{\mathrm{d}t} \propto \sqrt{y}$$

where y is the depth of the water above the plughole. In this particular problem, we know that $V = Ay$ where A is the area of the base of the cylinder; we don't actually need the value of A, but this equation says that $V \propto y$, hence

$$\frac{\mathrm{d}V}{\mathrm{d}t} \propto \frac{\mathrm{d}y}{\mathrm{d}t}$$

so

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -k\sqrt{y}$$

This ODE is separable:

$$y^{-\frac{1}{2}}\mathrm{d}y = -k\mathrm{d}t$$

Integration gives

$$2y^{\frac{1}{2}} = C - kt$$

We have two parameters, so we must use two measurements to nail them down:

$$2 \cdot 3 = 2 \cdot 9^{\frac{1}{2}} = C - 0 \quad \Rightarrow \quad C = 6$$
$$2 \cdot 2 = 2 \cdot 4^{\frac{1}{2}} = 6 - k \quad \Rightarrow \quad k = 2$$

Last, solve the equation $y(t) = 0$:

$$0 = 6 - 2t \quad \Rightarrow \quad t = 3$$

The tank is drained at 3 pm. □

Problem 3 (1.5.15)**.** Find a general solution to the ODE

$$\frac{\mathrm{d}y}{\mathrm{d}x} + 2xy = x$$

subject to the initial condition $y(0) = -2$.

Solution. The very observant reader might notice that this ODE is separable too, but we'll let that reader work through the solution from that direction.

The exponential coefficient is $e^{\int 2x\,\mathrm{d}x} = e^{x^2}$: we get

$$\begin{aligned} e^{x^2}\frac{\mathrm{d}y}{\mathrm{d}x} + ye^{x^2}2x &= xe^{x^2} \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(e^{x^2}y\right) &= xe^{x^2} \\ e^{x^2}y &= \int xe^{x^2}\,\mathrm{d}x = \frac{1}{2}e^{x^2} + C \\ y &= \frac{1}{2} + Ce^{-x^2} \end{aligned}$$

The initial condition gives

$$-2 = \frac{1}{2} + C \quad \Rightarrow \quad C = -\frac{5}{2}$$

□

Problem 4 (1.5.36)**.** A tank containing (initially) 60 gallons of distilled water has salt solution (2 gal/min of 1 lb/gal) flowing into the top and thoroughly mixed solution (3 gal/min) flowing out the bottom.

(a) Write the function $x(t)$ giving the amount (in lb.) of salt in the tank after t minutes.
(b) What is the greatest amount of salt in the tank at one time?

Solution. Many students spent a few lines showing how to use infinitesimals to get the ODE which models this problem. That kind of thinking is wonderful practice for when you are facing an entirely new situation which you suspect will need a differential equation to model. (That analysis was, however, not mandatory, since the text gave us the model

$$\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{r_{\text{out}}}{V}x = r_{\text{in}}c_{\text{in}}$$

where $x(t)$ is the weight (in lbs) of salt in the tank at time t, $V = V(t)$ is the volume of solution in the tank, and c_{in}, r_{in}, and r_{out} are the (constant) concentration of the inflowing solution (lb per gallon) and rates of in- and out-flow (in gallons per minute) respectively. Substituting in the known quantities, we have

$$\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{3}{60-t}x = 2$$

This is a linear ODE; our exponential coefficient function is

$$e^{\int \frac{3}{60-t}\mathrm{d}t} = e^{-3\log(60-t)} = (60-t)^{-3}$$

Multiplying through,

$$\begin{aligned}
(60-t)^{-3}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{3}{(60-t)^4}x &= 2(60-t)^{-3} \\
\frac{\mathrm{d}}{\mathrm{d}t}\left((60-t)^{-3}x\right) &= 2(60-t)^{-3} \\
(60-t)^{-3}x &= 2\int (60-t)^{-3}\mathrm{d}t = 2\cdot\frac{1}{+2}(60-t)^{-2} + C \\
x &= (60-t) + C(60-t)^3
\end{aligned}$$

Using the initial condition $x(0) = 0$,

$$1 + C(60-0)^2 = 0 \quad \Rightarrow \quad C = 60^{-2}$$

This completes part (a).

For part (b), we need to find a critical point of the function $x(t)$:

$$\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}t}_{|t_m} = 0 \text{ if} &\quad \frac{3}{60^2}(60-t_m)^2 - 1 = 0 \\
\text{if} &\quad (60-t_m)^2 = \frac{60^2}{3} \\
\text{if} &\quad 60 - t_m = \frac{60}{\sqrt{3}} \\
\text{if} &\quad t_m = 60(1-\frac{1}{\sqrt{3}}) \approx 25.36
\end{aligned}$$

A few students stopped here, forgetting that we need both the critical input and its output: $x(t_m) \approx 23.09$ lb. □