**VANDERBILT UNIVERSITY**
**MATH 196 — DIFFERENTIAL EQUATIONS WITH LINEAR ALGEBRA**
**EXAMPLES OF SECTIONS 7.5.**

Here we will look only at the case of eigenvalues with multiplicity two and defect one. Recall that this means that we have an eigenvalue $\lambda$ which is a double root of the characteristic equation

$$\det(A - \lambda I) = 0,$$

but such that, when we attempt to solve

$$(A - \lambda I)\vec{v} = \vec{0}, \tag{1}$$

we find only one linearly independent eigenvector. Since the multiplicity of $\lambda$ is two, we need two linearly independent eigenvectors associated wtih $\lambda$. Recall that for this to be the case, the system should have two free variable.

When we have such a missing eigenvector, we proceed as follows. Let $\vec{v}_1$ be an eigenvector that we found solving (1). Then find a solution $\vec{v}_2$ of

$$(A - \lambda I)^2 \vec{v}_2 = \vec{0}, \tag{2}$$

satisfying

$$(A - \lambda I)\vec{v}_2 = \vec{v}_1. \tag{3}$$

Notice that (3) may not be automatically satisfied, in which case we have to choose the free variables of (2) appropriately.

Then

$$\vec{x}_1 = \vec{v}_1 e^{\lambda t}$$

and

$$\vec{x}_2 = (t\vec{v}_1 + \vec{v}_2)e^{\lambda t}$$

are two linearly independent solutions associated with $\lambda$.

As an example, consider

$$\vec{x}' = \begin{bmatrix} 1 & -3 \\ 3 & 7 \end{bmatrix} \vec{x}.$$

The characteristic equation is

$$\det \begin{bmatrix} 1-\lambda & -3 \\ 3 & 7-\lambda \end{bmatrix} = (1-\lambda)(7-\lambda) + 9 = \lambda^2 - 8\lambda + 16 = (\lambda - 4)^2 = 0.$$

The solution is $\lambda = 4$, counted with multiplicity 2. Let us find the corresponding eigenvectors.

$$A - \lambda I = \begin{bmatrix} 3-\lambda & -3 \\ 3 & 7-\lambda \end{bmatrix} = \begin{bmatrix} -3 & -3 \\ 3 & 3 \end{bmatrix},$$

hence we want to solve

$$\begin{bmatrix} -3 & -3 \\ 3 & 3 \end{bmatrix} \vec{v}_1 = \begin{bmatrix} -3 & -3 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

We see that the first and second equations, $-3a - 3b = 0$ and $3a + 3b = 0$, are multiples of each other, so we have only one free variable. We find

$$\vec{v}_1 = a \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Dropping $a$ (or, more precisely, choosing $a = 1$),

$$\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Next, compute

$$(A - \lambda I)^2 = \begin{bmatrix} -3 & -3 \\ 3 & 3 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

and solve (2), i.e.,

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Any values of $a$ and $b$ solve this system, so we could choose $a = 1$, $b = 0$ and set

$$\vec{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

However, recall that we also need to satisfy (3). Computing, we find

$$(A - \lambda)\vec{v}_2 = \begin{bmatrix} -3 & -3 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \end{bmatrix} \neq \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \vec{v}_1.$$

But recall that we have freedom to choose the free variables, so if instead of $a = 1$ in the solution $\vec{v}_1$ we had chosen $a = -3$, then

$$\vec{v}_1 = \begin{bmatrix} -3 \\ 3 \end{bmatrix}$$

and (3) is satisfied (of course, we could instead keep $\vec{v}_1 = (1, -1)$ and choose $a = -\frac{1}{3}$, $b = 0$, for $\vec{v}_2$).

One solution is then

$$\vec{x}_1 = \vec{v}_1 e^{\lambda t} = \begin{bmatrix} -3 \\ 3 \end{bmatrix} e^{4t},$$

and another (linearly independent) one is

$$\vec{x}_2 = (t\vec{v}_1 + \vec{v}_2)e^{\lambda t} = \left(t \begin{bmatrix} -3 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) e^{4t}.$$

Let us verify that $\vec{x}_2$ is indeed a solution. Write

$$\vec{x}_2 = \left(t \begin{bmatrix} -3 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) e^{4t} = \begin{bmatrix} (-3t+1)e^{4t} \\ 3te^{4t} \end{bmatrix}.$$

Differentiating,

$$\vec{x}_2' = \begin{bmatrix} \left((-3t+1)e^{4t}\right)' \\ \left(3te^{4t}\right)' \end{bmatrix} = \begin{bmatrix} -3e^{4t} + 4(-3t+1)e^{4t} \\ 3e^{4t} + 12te^{4t} \end{bmatrix} = \begin{bmatrix} 1 - 12t \\ 3 + 12t \end{bmatrix} e^{4t}.$$

On the other hand,

$$A\,\vec{x}_2 = \begin{bmatrix} 1 & -3 \\ 3 & 7 \end{bmatrix} \begin{bmatrix} (-3t+1)e^{4t} \\ 3te^{4t} \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 3 & 7 \end{bmatrix} \begin{bmatrix} -3t+1 \\ 3t \end{bmatrix} e^{4t}$$
$$= \begin{bmatrix} -3t+1-3\times 3t \\ 3(-3t+1)+7\times 3t \end{bmatrix} e^{4t} = \begin{bmatrix} 1-12t \\ 3+12t \end{bmatrix} e^{4t}.$$

Hence

$$\vec{x}_2' = \begin{bmatrix} 1-12t \\ 3+12t \end{bmatrix} e^{4t} = A\,\vec{x}_2 = \begin{bmatrix} 1-12t \\ 3+12t \end{bmatrix} e^{4t}$$

as desired.