VANDERBILT UNIVERSITY
MATH 196 — DIFFERENTIAL EQUATIONS WITH LINEAR ALGEBRA
EXAMPLES OF SECTION 5.5.

Question 1. Write the form of the particular solution for the equations below (you do not have to find the values of the constants).
(a) $y'' + 9y = 2\cos(3x) + 3\sin(3x)$.
(b) $y''' + y' = 2 - \sin x$.
(c) $y^{(4)} - 2y'' + y = xe^x$.
(d) $y'' + 9y = 2x^2e^{3x} + 5$.

Question 2. In the questions below, two linearly independent solutions of the associated homogeneous equation are given. Find the particular solution y_p.
(a) $x^2y'' - 4xy' + 6y = x^3$, $y_1 = x^2$, $y_2 = x^3$.
(a) $(x^2 - 1)y'' - 2xy' + 2y = x^2 - 1$, $y_1 = x$, $y_2 = 1 + x^2$.

Question 3. Show that the formula given in class for y_p in fact produces a particular solution.

SOLUTIONS.

1a. The homogeneous equation is

$$y'' + 9y = 0,$$

with characteristic equation

$$\lambda^2 + 9 = 0,$$

whose roots are $\pm 3i$. Hence

$$y_1 = \cos(3x),\ y_2 = \sin(3x),$$

are solutions of the homogeneous equation. Given the form of $f(x)$, we look for

$$y_p = x^s\big(A\cos(3x) + B\sin(3x)\big).$$

Since $\cos(3x)$ and $\sin(3x)$ are solutions of the homogeneous equation, we need $s = 1$, so

$$y_p = x\big(A\cos(3x) + B\sin(3x)\big).$$

1b. The homogeneous equation is

$$y''' + y' = 0,$$

with characteristic equation

$$\lambda^3 + \lambda = 0,$$

whose roots are $\lambda = 0$ and $\lambda = \pm i$. Hence

$$y_1 = 1,\ y_2 = \cos(x),\ y_3 = \sin(x),$$

are solutions of the homogeneous equation. Given the form of $f(x)$, we look for

$$y_p = x^sA + x^r\big(B\cos(x) + C\sin(x)\big),$$

where A corresponds to 2 and $B\cos(x) + C\sin(x)$ to $\sin(x)$. If $s = 0$ then A =constant repeats the solution y_1, while if $r = 0$ then $\cos(x)$ and $\sin(x)$ repeat y_2 and y_3. Hence we set $s = 1$ and $r = 1$:

$$y_p = Ax + x\big(B\cos(3x) + C\sin(x)\big).$$

1c. The homogeneous equation is

$$y^{(4)} - 2y'' + y = 0,$$

with characteristic equation

$$\lambda^4 - 2\lambda^2 + 1 = 0,$$

whose roots are ± 1, each repeated twice. Hence

$$y_1 = e^{-x},\ y_2 = xe^{-x},\ y_3 = e^x,\ y_4 = xe^x,$$

are solutions of the homogeneous equation. Given the form of $f(x)$, we look for

$$y_p = x^s\big(Ax + B\big)e^x.$$

Since e^x and xe^x are solutions of the homogeneous equation, we need $s = 2$, so

$$y_p = x^2\big(Ax + B\big)e^x.$$

1d. The homogeneous equation is

$$y'' + 9y = 0,$$

with characteristic equation

$$\lambda^2 + 9 = 0,$$

whose roots are $\pm 3i$. Hence

$$y_1 = \cos(3x),\ y_2 = \sin(3x),$$

are solutions of the homogeneous equation. Given the form of $f(x)$, we look for

$$y_p = x^s A + x^r\big(Bx^2 + Cx + D\big)e^{3x}.$$

Since there is no repetition with the solutions of the homogeneous equation, $r = s = 0$ and

$$y_p = A + \big(Bx^2 + Cx + D\big)e^{3x}.$$

For question 2, recall that the formula for y_p is

$$y_p(x) = -y_1(x)\int \frac{y_2(x)f(x)}{W(x)}\,dx + y_2(x)\int \frac{y_2(x)f(x)}{W(x)}\,dx.$$

But it is very important to also remember that this formula is for an equation written with the coefficient of y'' being one. If this is not the case we have to first divide the equation by the coefficient of y''.

2a. Rewrite the equation as

$$y'' - \frac{4}{x}y' + \frac{6}{x^2}y = x,$$

so that $f(x) = x$. Compute

$$W(x) = y_1y_2' - y_2y_1' = x^2(3x^2) - x^3(2x) = x^4.$$

Plugging all quantities into the formula for y_p and performing the integrals we find

$$y_p = x^3(\ln x - 1).$$

2b. Rewrite the equation as

$$y'' - \frac{2x}{x^2-1}y' + \frac{2}{x^2-1}y = 1$$

so that $f(x) = 1$. Compute

$$W(x) = y_1 y_2' - y_2 y_1' = x(2x) - (1+x^2)1 = x^2 - 1.$$

Plugging all quantities into the formula for y_p and performing the integrals we find

$$y_p = -x^2 + x\ln\left|\frac{1+x}{1-x}\right| + \frac{1}{2}(1+x^2)\ln|1-x^2|.$$

3. We want to show that

$$y_p(x) = -y_1(x)\int \frac{y_2(x)f(x)}{W(x)}\,dx + y_2(x)\int \frac{y_2(x)f(x)}{W(x)}\,dx.$$

solves

$$y'' + py' + qy = f,$$

where p, q and f are not necessarily constants and y_1 and y_2 are solutions of the associated homogeneous equation, i.e., they solve

$$y'' + py' + qy = 0.$$

We have to plug y_p into the equation. Since we shall differentiate y_p, it is useful to remember the Fundamental Theorem of Calculus, which gives

$$\Big(\int \frac{y_2(x)f(x)}{W(x)}\,dx\Big)' = \frac{y_2(x)f(x)}{W(x)},$$

and

$$\Big(\int \frac{y_2(x)f(x)}{W(x)}\,dx\Big)' = \frac{y_2(x)f(x)}{W(x)}.$$

Using these formulas and the product rule we find

$$y_p' = -y_1'\int \frac{y_2 f}{W} - y_1\frac{y_2 f}{W} + y_2'\int \frac{y_1 f}{W} + y_2\frac{y_1 f}{W},$$

where we write $\int \frac{y_2 f}{W}$ instead of $\int \frac{y_2(x)f(x)}{W(x)}\,dx$ in order to simplify the notation (analogously for the other integral). Taking another derivative

$$y_p'' = -y_1''\int \frac{y_2 f}{W} - 2y_1'\frac{y_2 f}{W} - y_1\left(\frac{y_2 f}{W}\right)' + y_2''\int \frac{y_1 f}{W} + 2y_2'\frac{y_1 f}{W} + y_2\left(\frac{y_1 f}{W}\right)'.$$

Using y_p, y_p' and y_p'' into the equation we find

$$\begin{aligned} y_p'' + py_p' + qy_p = &-(y_1'' + py_1' + qy_1)\int \frac{y_2 f}{W} + (y_2'' + py_2' + qy_2)\int \frac{y_2 f}{W} \\ &- \frac{y_2 f}{W}(py_1 + 2y_1') + \frac{y_1 f}{W}(py_2 + 2y_2') - y_1\left(\frac{y_2 f}{W}\right)' + y_2\left(\frac{y_1 f}{W}\right)'. \end{aligned}$$

Since by hypothesis y_1 and y_2 are solutions of the homogeneous equation,

$$y_1'' + py_1' + qy_1 = 0,$$

and

$$y_2'' + py_2' + qy_2 = 0,$$

so

$$y_p'' + py_p' + qy_p = -\frac{y_2 f}{W}(py_1 + 2y_1') + \frac{y_1 f}{W}(py_2 + 2y_2') - y_1\left(\frac{y_2 f}{W}\right)' + y_2\left(\frac{y_1 f}{W}\right)'$$

By the product rule

$$\left(\frac{y_2 f}{W}\right)' = y_2'\frac{f}{W} + y_2 f'\frac{1}{W} + y_2 f\left(\frac{1}{W}\right)',$$

and

$$\left(\frac{y_1 f}{W}\right)' = y_1'\frac{f}{W} + y_1 f'\frac{1}{W} + y_1 f\left(\frac{1}{W}\right)'.$$

Therefore

$$\begin{aligned} y_p'' + py_p' + qy_p = &-\frac{y_2 f}{W}(py_1 + 2y_1') + \frac{y_1 f}{W}(py_2 + 2y_2') \\ &- y_1 y_2'\frac{f}{W} - y_1 y_2 f'\frac{1}{W} - y_1 y_2 f\left(\frac{1}{W}\right)' \\ &+ y_2 y_1'\frac{f}{W} + y_2 y_1 f'\frac{1}{W} + y_2 y_1 f\left(\frac{1}{W}\right)'. \end{aligned}$$

Notice that the last two terms of the second line cancel with the last two terms of the third line. We are left with

$$\begin{aligned} y_p'' + py_p' + qy_p &= -\frac{y_2 f}{W}(py_1 + 2y_1') + \frac{y_1 f}{W}(py_2 + 2y_2') - y_1 y_2'\frac{f}{W} + y_2 y_1'\frac{f}{W} \\ &= -p\frac{y_2 f}{W}y_1 - 2\frac{y_2 f}{W}y_1' + p\frac{y_1 f}{W}y_2 + 2\frac{y_1 f}{W}y_2' - y_1 y_2'\frac{f}{W} + y_2 y_1'\frac{f}{W}. \end{aligned}$$

The first and third terms on the last line cancel out, and then

$$\begin{aligned} y_p'' + py_p' + qy_p &= -2\frac{y_2 f}{W}y_1' + 2\frac{y_1 f}{W}y_2' - y_1 y_2'\frac{f}{W} + y_2 y_1'\frac{f}{W} \\ &= -2\frac{f}{W}y_2 y_1' + 2\frac{f}{W}y_1 y_2' - y_1 y_2'\frac{f}{W} + y_2 y_1'\frac{f}{W} \\ &= \frac{f}{W}y_1 y_2' - \frac{f}{W}y_2 y_1' = \frac{f}{W}(y_1 y_2' - y_2 y_1') \\ &= f, \end{aligned}$$

where in the last step we used that $W = y_1 y_2' - y_2 y_1'$.