VANDERBILT UNIVERSITY
MATH 196 — DIFFERENTIAL EQUATIONS WITH LINEAR ALGEBRA
EXAMPLES OF SECTIONS 5.2 AND 5.3.

Question 1. Find the general solution of

$$y''' - 5y'' + 8y' - 4y = 0.$$

Question 2. Find the general solution of

$$3y''' - 2y'' + 12y' - 8y = 0,$$

knowing that $y = e^{\frac{2}{3}x}$ is a solution.

SOLUTIONS.

1. The characteristic equation is

$$\lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0.$$

This can be factored as

$$(\lambda - 1)(\lambda^2 - 4\lambda + 4) = 0,$$

which factors further into

$$(\lambda - 1)(\lambda - 2)^2 = 0.$$

Hence the roots are $\lambda_1 = 1$ and $\lambda_2 = 2$, with this last solution counted with multiplicity two. Following the rules for construction of solutions of homogeneous equations with constant coefficients seen in class (see also in the textbook: Theorem 1, p. 315; Theorem 2, p. 318; Theorem 3, p. 320; and the explanation on p. 322), one finds

$$\begin{aligned} y_1 &= e^{\lambda_1 x} = e^x, \\ y_2 &= e^{\lambda_2 x} = e^{2x}, \\ y_3 &= xe^{\lambda_2 x} = xe^{2x}, \end{aligned}$$

so that the general solution is

$$y = c_1 e^x + c_2 e^{2x} + c_3 x e^{2x}.$$

Remark. If you do not see right away how to factor the polynomial $\lambda^3 - 5\lambda^2 + 8\lambda - 4$, remember the following trick. When given a polynomial of this form — i.e., integer coefficients and a zeroth order term a_0 which does not contain the variable λ —, try plugging in the divisors of a_0 into the equation and see if one of them is a root. In our example, $a_0 = -4$, so we try ± 1, ± 2 and ± 4. We see then that 1 is a root of $\lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$, what implies that $\lambda - 1$ *divides* the polynomial $\lambda^3 - 5\lambda^2 + 8\lambda - 4$. Doing long division of polynomials (remember your high school algebra, or division of polynomials when you learned integration by partial fractions), you find

$$\frac{\lambda^3 - 5\lambda^2 + 8\lambda - 4}{\lambda - 1} = \lambda^2 - 4\lambda + 4,$$

which is the same as

$$\lambda^3 - 5\lambda^2 + 8\lambda - 4 = (\lambda - 1)(\lambda^2 - 4\lambda + 4).$$

Now you can go ahead and factor $\lambda^2 - 4\lambda + 4$. Notice that this procedure also applies to polynomials of higher degree. Finally, if the lower order term does contain λ, that means that $\lambda = 0$ is a root, so you can first factor it and then apply the above procedure, e.g.

$$\lambda^5 - 5\lambda^4 + 8\lambda^3 - 4\lambda^2 = \lambda^2(\lambda^3 - 5\lambda^2 + 8\lambda - 4) = \lambda^2(\lambda - 1)(\lambda - 2)^2.$$

2. The characteristic equation is

$$3\lambda^3 - 2\lambda^2 + 12y\lambda - 8 = 0.$$

Since $y = e^{\frac{2}{3}x}$ is a solution, this means that $\lambda = \frac{2}{3}$ is a root of the characteristic equation. Hence, onc of the factors of the polynomial is $\lambda - \frac{2}{3}$ or, after multiplying by 3, $3\lambda - 2$. Performing long division,

$$\frac{3\lambda^3 - 2\lambda^2 + 12\lambda - 8}{3\lambda - 2} = \lambda^2 + 4,$$

or

$$3\lambda^3 - 2\lambda^2 + 12\lambda - 8 = (3\lambda - 2)(\lambda^2 + 4).$$

$\lambda^2 + 4$ gives $\pm 2i$, so we have the two solutions $\cos(2x)$ and $\sin(2x)$. The general solution is then

$$y = c_1 e^{\frac{2}{3}x} + c_2 \cos(2x) + c_3 \sin(2x).$$