VANDERBILT UNIVERSITY
MATH 196 — DIFFERENTIAL EQUATIONS WITH LINEAR ALGEBRA
EXAMPLES OF SECTIONS 4.5 AND 4.6.

In the problems below, let A be the matrix

$$A = \begin{bmatrix} 1 & -4 & -3 & -7 \\ 2 & -1 & 1 & 7 \\ 1 & 2 & 3 & 11 \end{bmatrix}.$$

Question 1. Give the column and row spaces of A in terms of a basis.

Question 2. What is the dimension of $\ker(A)$?

Question 3. Find a basis for $\ker(A)$ by computing the orthogonal complement to $Row(A)$.

Solutions.

1. Applying Gauss-Jordan elimination we find

$$rref(A) = \begin{bmatrix} 1 & 0 & 1 & 5 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The first two columns are pivot columns, i.e., they contain a leading one. Therefore the first two columns of A are linearly independent, and

$$Col(A) = \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -4 \\ -1 \\ 2 \end{bmatrix} \right\}.$$

The non-zero rows of $rref(A)$ are the first and the second, therefore

$$Row(A) = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 3 \end{bmatrix} \right\}.$$

Remark. It is important to remember that, after finding $rref(A)$, the columns that form a basis of $Col(A)$ are the columns of the original matrix (i.e., A itself, as opposed to $rref(A)$) which correspond to pivot columns, while a basis for $Row(A)$ is given by the non-zero rows of $rref(A)$ — and not of the original matrix A.

2. Recall that

$$\text{rank} + \dim\ker(A) = \#\text{ of columns }.$$

Since $rref(A)$ has two leading ones, its rank is 2, hence $\dim\ker(A) = 2$.

3. Denote by $\vec{u}$ and $\vec{v}$ the vectors forming a basis for $Row(A)$ found in problem 1. If $\vec{x} = (x_1, x_2, x_3, x_4)$ belongs to the kernel of A, then

$$\langle \vec{u}, \vec{x} \rangle = 0$$

and

$$\langle \vec{v}, \vec{x} \rangle = 0.$$

Computing

$$\langle \vec{u}, \vec{x} \rangle = x_1 + x_3 + 5x_4 = 0$$

and

$$\langle \vec{v}, \vec{x} \rangle = x_2 + x_3 + 3x_4 = 0.$$

From these two equations we get

$$x_1 = -x_3 - 5x_4,$$
$$x_2 = -x_3 - 3x_4.$$

Since x_3 and x_4 are free variables, we can denote them by $x_3 = s$, $x_4 = t$, and write

$$x_1 = -s - 5t,$$
$$x_2 = -s - 3t.$$

Therefore $\vec{x} = (x_1, x_2, x_3, x_4)$ can be written as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -s - 5t \\ -s - 3t \\ s \\ t \end{bmatrix} = s \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -5 \\ -3 \\ 0 \\ 1 \end{bmatrix}.$$

The vectors

$$\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} -5 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

form a basis of $\ker(A)$.