VANDERBILT UNIVERSITY
MATH 196 — DIFFERENTIAL EQUATIONS WITH LINEAR ALGEBRA
EXAMPLES OF SECTIONS 1.6.

Solve the following IVP.

(a)

$$\begin{cases} xy' + (2x-3)y = 5x^5y^4, \\ y(1) = 1. \end{cases}$$

(b)

$$\begin{cases} (4xy + 6y^2)y' + 3x^2 + 2y^2 = 0, \\ y(0) = 2. \end{cases}$$

(c)

$$\begin{cases} y'' = 2yy', \\ y(1) = 1,\ y'(1) = 2. \end{cases}$$

SOLUTIONS.

(a). Write the equation as

$$y' + \frac{2x-3}{x}y = 5x^4y^4,$$

for $x \neq 0$, which is a Bernoulli equation with $n = 4$. Set $v = y^{1-n} = y^{-3}$. The equation for v then becomes

$$\frac{dv}{dx} + (1-4)\frac{2x-3}{x}v = (1-4)5x^4,$$

or

$$\frac{dv}{dx} + \Big(\frac{9}{x} - 6\Big)v = -15x^4.$$

Using the formula for first order linear D.E.'s with $p(x) = \frac{9}{x} - 6$ and $q(x) = -15x^4$, we have

$$e^{-\int p(x)\,dx} = e^{6x - 9\ln x} = x^{-9}e^{6x},$$
$$e^{\int p(x)\,dx} = e^{-6x + 9\ln x} = x^9e^{-6x}.$$

where we assumed $x > 0$ since the problem is defined only for $x > 0$ or $x < 0$ (because $x \neq 0$). From this we get

$$\int q(x)e^{\int p(x)\,dx}\,dx = -15\int x^{13}e^{-6x}\,dx.$$

This integral is done by a tiresome (but not difficult) process of integrating by parts thirteen times. The answer is

$$-\frac{e^{-6x}}{314928}\Big(25025 + 150150x + 450450x^2 + 900900x^3 + 1351350x^4 + 1621620x^5 + 1621620x^6$$

$$+ 1389960x^7 + 1042470x^8 + 694980x^9 + 416988x^{10} + 227448x^{11} + 113724x^{12} + 52488x^{13}\Big)$$

Denote the above expression by $f(x)$. Then using the formula for solutions of first order linear equations,

$$v(x) = x^{-9}e^{6x}\Big(f(x) + C\Big),$$

from which follows

$$y(x) = \Big[x^{-9}e^{6x}\Big(f(x) + C\Big)\Big]^{-\frac{1}{3}}.$$

To find C use $y(1) = 1$,

$$y(1) = 1 = \Big[e^{6}\Big(f(1) + C\Big)\Big]^{-\frac{1}{3}} \Rightarrow C = e^{-6} - f(1),$$

thus

$$y(x) = \Big[x^{-9}e^{6x}\Big(f(x) + e^{-6} - f(1)\Big)\Big]^{-\frac{1}{3}}.$$

Remark. Notice that despite its ugly form, the integral $\int x^{13}e^{-6x}\,dx$ is easy, in the sense that it can be done by a systematic application of integration by parts. In another words, students should be able to recognize the difference between problems which are long, but that can be done with enough patience via standard calculus techniques, from those problems which are genuinely difficult.

(b). Write the equation as

$$(4xy + 6y^2)dy + (3x^2 + 2y^2)dx = 0$$

Let $M(x, y) = 3x^2 + 2y^2$ and $N(x, y) = 4xy + 6y^2$. Computing we find $\frac{\partial M}{\partial y} = 4y$ and $\frac{\partial N}{\partial x} = 4y$, and therefore this is an exact equation since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. Put

$$F(x, y) = \int M(x, y)\,dx = \int (3x^2 + 2y^2)\,dx = x^3 + 2xy^2 + g(y).$$

Then

$$\frac{\partial F}{\partial y} = 4xy + g'(y) = N = 4xy + 6y^2 \Rightarrow g'(t) = 6y^2.$$

Integrating,

$$g(y) = 2y^3$$

(we do not need to add a constant here). Hence the general solution is

$$F(x, y) = x^3 + 2xy^2 + 2y^3 = C.$$

Using $y(0) = 2$ we find $C = 16$, so the solution to the IVP is

$$F(x, y) = x^3 + 2xy^2 + 2y^3 - 16 = 0.$$

(c). Make the substitution $v = y'$. Then

$$y'' = \frac{dv}{dx} = \frac{dv}{dy}\frac{dy}{dx} = \frac{dv}{dy}y' = \frac{dv}{dy}v,$$

which gives upon plugging into the original equation,

$$v\frac{dv}{dy} = 2yv \Rightarrow \frac{dv}{dy} = 2y.$$

Integrating,

$$v = y^2 + C$$

Since $v = y'$,

$$y' = y^2 + C.$$

From $y(1) = 1$, $y'(1) = 2$, we then have

$$y'(1) = (y(1))^2 + C \Rightarrow 2 = 1 + C \Rightarrow C = 1.$$

So

$$y' = y^2 + 1,$$

or

$$\frac{dy}{y^2+1} = dx.$$

Integrating

$$\arctan(y) = x + C \Rightarrow y = \tan(x + C).$$

Using again $y(1) = 1$:

$$1 = \tan(1 + C) \Rightarrow C = \frac{\pi}{4} - 1,$$

so

$$y = \tan(x + \frac{\pi}{4} - 1).$$