VANDERBILT UNIVERSITY
MATH 196 — DIFFERENTIAL EQUATIONS WITH LINEAR ALGEBRA
EXAMPLES OF SECTIONS 1.1 AND 1.2

Question 1. Write a differential equation modeling the described situation.
(a) The line tangent to the graph of a function $f(x)$ at the point (x, y) intersects the x-axis at the point $(\frac{x}{2}, 0)$.
(b) The rate of change of a population is proportional to the square root of the population.

Question 2. A projectile is fired straight upward with an initial velocity of $100\, m/s$ from the top of a building $20\, m$ high and falls to the ground at the base of the building. Find
(a) its maximum height above the ground;
(b) when it passes the top of the building;
(c) the total time in the air.

SOLUTIONS.

1a. The slope of the line through (x, y) and $(\frac{x}{2}, 0)$ is

$$\frac{y-0}{x-\frac{x}{2}} = 2\frac{y}{x}.$$

Thus

$$y' = 2\frac{y}{x}.$$

1b. We have

$$\frac{dP}{dt} \propto \sqrt{P} \Rightarrow \frac{dP}{dt} = k\sqrt{P},$$

where k is a constant.

2. The acceleration of gravity is $-9.8\, m/s$ with the y-axis oriented upward. Since gravity is the only force acting on the projectile,

$$a = \frac{dv}{dt} = -9.8 \Rightarrow \int dv = -9.8 \int dt \Rightarrow v = -9.8t + C.$$

But $v(0) = 100$ so

$$v = -9.8t + 100. \tag{1}$$

Integrate again to find the position y:

$$v = \frac{dy}{dt} = -9.8t + 100 \Rightarrow \int dy = \int (-9.8t + 100)dt \Rightarrow y = -4.9t^2 + 100t + C.$$

Since $y(0) = 20$, we obtain

$$y = -4.9t^2 + 100t + 20. \tag{2}$$

(a) At the maximum point, $v = 0$. Setting $v = 0$ in (1) gives $t = \frac{100}{9.8}$. Using this into (2) produces $y(\frac{100}{9.8}) = -4.9(\frac{100}{9.8})^2 + 100 \times \frac{100}{9.8} + 20 \approx 530$ meters.
(b) It passes the top of the building when $y(t) = -4.9t^2 + 100t + 20 = 20$, which gives two solutions, $t = 0$ (when the projectile is launched) and $t = \frac{100}{4.9} \approx 20.4$ seconds, which is the desired answer.
(c) It reaches the ground when $y = 0$. Solving $-4.9t^2 + 100t + 20 = 0$ yields $t = 20.61$ seconds and $t = -0.2$ seconds. The second solution is not physical, hence the answer is 20.61 seconds.