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11.4

<u>Comparison Test</u>: Suppose $\sum a_n$ and $\sum b_n$ have positive terms.

1.) If $\sum b_n$ is convergent, and $a_n \leq b_n$, then $\sum a_n$ converges.

2.) If $\sum b_n$ is divergent, and $a_n \geq b_n$, then $\sum a_n$ diverges.

<u>Example</u>: $\sum_{n=1}^{\infty} \frac{3^n}{2^n-5}$

$$\frac{3^n}{2^n-5} > \frac{3^n}{2^n} = \left(\frac{3}{2}\right)^n$$

Since $\sum_{n=1}^{\infty} \left(\frac{3}{2}\right)^n$ diverges (geometric series with $r=\frac{3}{2}$), by the Comparison Test, this series diverges as well.

<u>Example</u>: $\sum_{n=1}^{\infty} \frac{n}{n^4+n^3+2}$

$$\frac{n}{n^4+n^3+2} < \frac{n}{n^4} = \frac{1}{n^3}$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges (p-series with $p=3$), by the Comparison Test, this series converges as well.

<u>Limit Comparison Test</u>: Suppose $\sum a_n$ and $\sum b_n$ have positive terms. If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$, where c is positive and finite, then either both series converge or both series diverge.

<u>Example</u>: $\sum_{n=1}^{\infty} \frac{10^{n+1}}{11^n-n}$

Compare the terms to $\left(\frac{10}{11}\right)^n$.

$$\lim_{n\to\infty} \frac{\left(\frac{10^{n+1}}{11^n-n}\right)}{\left(\frac{10^n}{11^n}\right)} = \lim_{n\to\infty} \frac{10}{1-\frac{n}{11^n}} = 10 \quad \left(\lim_{x\to\infty} \frac{x}{11^x} \overset{L'H}{=} \lim_{x\to\infty} \frac{1}{(\ln 11)11^x} = 0\right)$$

Since $\sum_{n=1}^{\infty} \left(\frac{10}{11}\right)^n$ converges (geometric series with $r=\frac{10}{11}$), by the Limit Comparison Test, this series converges as well.

Example: $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{4n^3-3n+1}}{n^2+6n}$

Compare the terms to $\frac{1}{\sqrt{n}}$.

$$\lim_{n\to\infty} \frac{\left(\frac{\sqrt{4n^3-3n+1}}{n^2+6n}\right)}{\left(\frac{1}{\sqrt{n}}\right)} = \lim_{n\to\infty} \frac{\sqrt{4n^4-3n^2+n}}{n^2+6n} = \lim_{n\to\infty} \frac{\sqrt{4-\frac{3}{n^2}+\frac{1}{n^3}}}{1+\frac{6}{n}} = 2$$

Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges (p-series with $p=\frac{1}{2}$), by the Limit Comparison Test, this series diverges as well.

11.5

Example: $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots = \sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{1}{n}$

$s_2 = 1 - \frac{1}{2} = \frac{1}{2}$

$s_4 = s_2 + \left(\frac{1}{3} - \frac{1}{4}\right) = \frac{1}{2} + \frac{1}{12} = \frac{7}{12}$

$s_6 = s_4 + \left(\frac{1}{5} - \frac{1}{6}\right) = \frac{7}{12} + \frac{1}{30} = \frac{37}{60}$

$\vdots$

$s_{2n+2} = s_{2n} + \left(\frac{1}{2n+1} - \frac{1}{2n+2}\right) = s_{2n} + \frac{1}{(2n+1)(2n+2)}$

$s_2 \leq s_4 \leq s_6 \leq \dots \leq s_{2n} \leq s_{2n+2} \leq \dots$

On the other hand, $s_{2n} = 1 - \left(\frac{1}{2} - \frac{1}{3}\right) - \left(\frac{1}{4} - \frac{1}{5}\right) - \dots - \left(\frac{1}{2n-2} - \frac{1}{2n-1}\right) - \frac{1}{2n} < 1.$

Thus, $\lim_{n \to \infty} s_{2n} = s \leq 1.$

$\lim_{n \to \infty} s_{2n+1} = \lim_{n \to \infty} \left(s_{2n} + \frac{1}{2n+1}\right) = s$

Therefore, the series converges to s. [Alternating Series Test]

Alternating Series Test: If $\sum_{n=1}^{\infty} (-1)^n b_n$, where $b_n > 0$, satisfies

i.) $\lim_{n \to \infty} b_n = 0$, and ii.) $b_n \geq b_{n+1}$, then the series converges.