VANDERBILT UNIVERSITY
MAT 155B, FALL 12 — TEST 3 SOLUTIONS

Question 1 [5 pts]. Solve the differential equation

$$y' = y^2 \sin x.$$

Solution.

$$\begin{aligned} \frac{dy}{dx} &= y^2 \sin x \\ \Rightarrow \int \frac{dy}{y^2} &= \int \sin x \, dx \\ \Rightarrow -\frac{1}{y} &= -\cos x + C \Rightarrow y = \frac{1}{\cos x - C}. \end{aligned}$$

$y = 0$ is also a solution.

Question 2 [10 pts]. Solve the initial value problem

$$\begin{cases} y' = 6x^2y^2, \\ y(0) = 1. \end{cases}$$

Solution.

$$\begin{aligned} \frac{dy}{dx} &= 6x^2y^2 \\ \Rightarrow \int \frac{dy}{y^2} &= 6\int x^2 \, dx \\ \Rightarrow -\frac{1}{y} &= 2x^3 + C \Rightarrow y = \frac{1}{-2x^3 + C}. \end{aligned}$$

Plugging $y(0) = 1$ we find $y = \frac{1}{1-2x^3}$.

Question 3 [15 pts]. Determine whether each of the following sequences converges or diverges. When it converges, find its limit.

(a) [5 pts]. $a_n = \dfrac{n^2 - 4}{n^2 + 2n}$

Solution. Converges to 1:

$$\lim_{n\to\infty} \frac{n^2 - 4}{n^2 + 2n} = \lim_{n\to\infty} \frac{1 - \frac{4}{n^2}}{1 + \frac{2}{n}} = 1.$$

(b) [5 pts]. $a_n = \arctan(n!)$

Solution. Converges to $\frac{\pi}{2}$:

$$\lim_{n\to\infty} \arctan(n!) = \arctan(\lim_{n\to\infty} n!) = \arctan(\infty) = \frac{\pi}{2}.$$

(c) [5 pts]. $a_1 = 2$, $a_{n+1} = a_n + \dfrac{1}{n}$

Solution. Diverges. Notice that

$$\begin{aligned}
a_{n+1} = a_n + \frac{1}{n} = a_{n-1} + \frac{1}{n-1} + \frac{1}{n} &= a_{n-2} + \frac{1}{n-2} + \frac{1}{n-1} + \frac{1}{n} \\
&= a_{n-3} + \frac{1}{n-3} + \frac{1}{n-2} + \frac{1}{n-1} + \frac{1}{n} \\
&= \cdots \\
&= a_1 + \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-3} + \frac{1}{n-2} + \frac{1}{n-1} + \frac{1}{n} \\
&= 2 + \sum_{i=1}^{n} \frac{1}{i} \geq \sum_{i=1}^{n} \frac{1}{i}.
\end{aligned}$$

Taking the limit, $\sum \frac{1}{i}$ becomes the harmonic series, which diverges, so the sequence diverges as well.

Question 4 [40 pts]. Determine whether each of the following series converges or diverges. You do not have to compute the sum if the series converges.

(a) [5 pts]. $\displaystyle\sum_{n=1}^{\infty} \frac{n+4}{n^2+5n+4}$

Solution. Compare with $\frac{1}{n}$

$$\lim_{n\to\infty} \frac{\frac{n+4}{n^2+5n+4}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{\frac{n+4}{(n+1)(n+4)}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{\frac{1}{n+1}}{\frac{1}{n}} = 1.$$

Since $\sum \frac{1}{n}$ diverges (harmonic series), $\sum \frac{n+4}{n^2+5n+4}$ diverges by the limit comparison test.

(b) [5 pts]. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$

Solution. $x(\ln x)^2$ is increasing for $x \geq 2$, so $\frac{1}{x(\ln x)^2}$ is decreasing and we can use the integral test with the u-substitution $u = \ln x$:

$$\int_2^{\infty} \frac{1}{x(\ln x)^2}\,dx = \int_{\ln 2}^{\infty} \frac{1}{u^2}\,du.$$

This is a p-integral with $p = 2$, so it converges. Hence the series converges by the integral test.

(c) [10 pts]. $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$

Solution. Since $\ln n \leq \sqrt{n}$ for $n \geq 1$,

$$\sum_{n=1}^{\infty} \frac{\ln n}{n^2} \leq \sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}.$$

The series on the right hand side is a p-series with $p = \frac{3}{2} > 1$, so it converges. Hence $\sum \frac{\ln n}{n^2}$ converges by comparison.

(d) [10 pts]. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\sqrt{n}}{n+4}$

Solution. Let us use the alternating series test. We have $b_n = \frac{\sqrt{n}}{n+4} \to 0$ as $n \to \infty$. To show that b_n is decreasing, consider

$$f(x) = \frac{\sqrt{x}}{x+4},$$

and take its derivative to find:

$$f'(x) = \frac{-x+4}{2\sqrt{x}(x+4)^2}.$$

So $f'(x) < 0$ for all $x > 4$, hence b_n is decreasing for $n > 4$, hence by the alternating series test, the series converges.

(e) [10 pts]. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+\ln n}$

Solution. Compare with $\frac{1}{n}$

$$\lim_{n\to\infty} \frac{\frac{1}{n+\ln n}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{n+\ln n}$$
$$= \frac{\infty}{\infty} \overset{L'H}{=} \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = 1.$$

Since $\sum \frac{1}{n}$ diverges (harmonic series), $\sum \frac{1}{n+\ln n}$ diverges by the limit comparison test.

Question 5 [20 pts]. Find the sum of the following convergent series.

(a) [10 pts]. $\displaystyle\sum_{n=1}^{\infty} e^{-n}$

Solution. This is a geometric series with $r = e^{-1} = \frac{1}{e}$ and missing the $n = 0$ term, so write

$$\sum_{n=1}^{\infty} e^{-n} = -1 + \sum_{n=0}^{\infty} e^{-n}.$$

Since $0 < e^{-1} < 1$,

$$\sum_{n=0}^{\infty} e^{-n} = \frac{1}{1-\frac{1}{e}} = \frac{e}{e-1},$$

so

$$\sum_{n=1}^{\infty} e^{-n} = -1 + \sum_{n=0}^{\infty} e^{-n} = -1 + \frac{e}{e-1} = \frac{1}{e-1}.$$

(b) [10 pts]. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+3n+2}$

Solution. Write

$$\frac{1}{n^2+3n+2} = \frac{1}{(n+1)(n+2)} = \frac{1}{n+1} - \frac{1}{n+2},$$

so

$$\sum_{n=1}^{\infty} \frac{1}{n^2+3n+2} = \sum_{n=1}^{\infty} \Big(\frac{1}{n+1} - \frac{1}{n+2}\Big).$$

This is telescoping series whose sequence of partial sums is

$$S_N = \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \frac{1}{4} - \cdots - \frac{1}{N+2} = \frac{1}{2} - \frac{1}{N+2}$$

Hence

$$\sum_{n=1}^{\infty} \Big(\frac{1}{n+1} - \frac{1}{n+2}\Big) = \lim_{N\to\infty} \Big(\frac{1}{2} - \frac{1}{N+2}\Big) = \frac{1}{2}.$$

Question 6 [10 pts]. How many terms of the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{2n^3}$$

do we need to add in order to find its sum correct to two decimal places? **Solution.** Use

$$b_{n+1} = \frac{1}{2(n+1)^3} \leq 0.01 = \frac{1}{100}$$
$$\Leftrightarrow (n+1)^3 \geq 50.$$

So we need at least $\sqrt[3]{50} - 1$ terms.

Extra credit [5 pts]. Consider the sequence given by $a_1 = 2$ and

$$a_{n+1} = 2 + \frac{1}{a_n}. \tag{1}$$

Determine if the argument below correct.

The limit of the sequence is $1 + \sqrt{2}$. In order to see this, notice that if

$$\lim_{n\to\infty} a_n = L,$$

then

$$\lim_{n\to\infty} a_{n+1} = L.$$

For example, for the sequence $c_n = \frac{1}{n}$, we have

$$\lim_{n\to\infty} c_n = \lim_{n\to\infty} \frac{1}{n} = 0,$$

and considering c_{n+1} we also find

$$\lim_{n\to\infty} c_{n+1} = \lim_{n\to\infty} \frac{1}{n+1} = 0,$$

so the limit of a_{n+1} is the same as that of a_n. Therefore, taking the limit on both sides of (1) gives

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \Big(2 + \frac{1}{a_n}\Big) \Longrightarrow L = 2 + \frac{1}{L}.$$

Solving for L then gives

$$L = 2 + \frac{1}{L} \iff L^2 - 2L - 1 = 0.$$

Using the quadratic formula,

$$L = \frac{-(-2) \pm \sqrt{(-2)^2 - 4 \times 1 \times (-1)}}{2} = \frac{2 \pm \sqrt{8}}{2} = 1 \pm \sqrt{2}.$$

Since the sequence is positive, we pick the positive root, so that $L = 1 + \sqrt{2}$, as desired.

Solution. The argument is incorrect in that it has never been showed that the sequence converges. Otherwise it would have been correct.