VANDERBILT UNIVERSITY
MAT 155B, FALL 12 — TAYLOR, MACLAURIN AND POWER SERIES IN A NUTSHELL.

1. Finding the radius of convergence of a power series.

Given a power series

$$\sum_{n=0}^{\infty} c_n (x-a)^n,$$

we find its radius of convergence by the following steps, which we will illustrate with the power series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{4^n(n+1)}(x-1)^n.$$

STEP 1. Put $a_n = c_n(x-a)^n$. In our example,

$$a_n = \frac{(-1)^n}{4^n(n+1)}(x-1)^n = \frac{(-1)^n(x-1)^n}{4^n(n+1)}.$$

STEP 2. Compute $\left|\frac{a_{n+1}}{a_n}\right|$, simplifying it as much as possible. In particular, the powers of n in $(x-a)$ will *always* simplify, and all the negative signs can be dropped:

$$\begin{aligned}\left|\frac{a_{n+1}}{a_n}\right| &= \left|\frac{\frac{(-1)^{n+1}(x-1)^{n+1}}{4^{n+1}(n+2)}}{\frac{(-1)^n(x-1)^n}{4^n(n+1)}}\right| \\ &= \left|\frac{(-1)^{n+1}(x-1)^{n+1}}{4^{n+1}(n+2)}\frac{4^n(n+1)}{(-1)^n(x-1)^n}\right| \\ &= \frac{n+1}{n+2}\frac{|x-1|}{4}.\end{aligned}$$

STEP 3. Take the limit $n \to \infty$ in the expression $\left|\frac{a_{n+1}}{a_n}\right|$ you found in step 2:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|.$$

If the limit is 0 (zero) then the radius of convergence if $R = \infty$. If the limit is ∞ (infinity) then the radius of convergence is $R = 0$. In either situation, we are done with the problem. If the limit is neither zero nor infinity, proceed to step 4.

STEP 4. We are taking the limit $n \to \infty$ in the expression $\left|\frac{a_{n+1}}{a_n}\right|$ found in step 2, and this is the case where the limit is neither zero nor infinity. So you will end up with an expression of the form

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L.$$

In our example,

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{n+1}{n+2}\frac{|x-1|}{4} = \frac{|x-1|}{4}.$$

Notice that after taking the limit, the resulting expression, $\frac{|x-1|}{4}$, has no n. This is *always the case:* the value L you find computing the limit no longer involves n.

STEP 5. Set $L < 1$ and solve for $|x - a|$. You will find an expression of the form

$$|x - a| < R,$$

where the number R on the right hand side is the answer, i.e., the radius of convergence. In our example,

$$\frac{|x-1|}{4} < 1 \Rightarrow |x-1| < 4.$$

Hence, the radius of convergence in this problem is $R = 4$.

A **common mistake** is the following. You *don't have* to solve for x in order to find the radius of convergence, i.e., you leave the expression in the form $x - a$. In the example, we found $|x - 1| < 4$: don't try to solve this for an inequality of the form $|x| < \cdots$. The only situation where the radius of convergence is going to be given by $|x| < R$ (with no a) is when $a = 0$.

2. Finding the interval of convergence of a power series.

Given a power series

$$\sum_{n=0}^{\infty} c_n(x-a)^n,$$

we find its interval of convergence by the following steps, which again we will illustrate with the power series

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{4^n(n+1)}(x-1)^n.$$

STEP 1. Find its radius of convergence as explained above. In our example we had $R = 4$.

STEP 2. Write $|x-a| = R$ and solve for x. You'll find two answers, and this will give you an interval of the form $(a - R, a + R)$,

$$|x-1| = 4 \Rightarrow x = 5,\ x = -3,$$

so that we have $(-3, 5)$.

STEP 3. Plug one of the values of x that you found into the power series. The resulting expression will be a series *with no* x, whose convergence can be analyzed by the earlier methods we learned.

Plugging $x = -3$ into

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{4^n(n+1)}(x-1)^n,$$

we find

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{(-1)^n}{4^n(n+1)}(-3-1)^n &= \sum_{n=0}^{\infty} \frac{(-1)^n}{4^n(n+1)}(-4)^n \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)}\left(\frac{-4}{4}\right)^n \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)}(-1)^n \\ &= \sum_{n=0}^{\infty} \frac{((-1)^n)^2}{(n+1)} \\ &= \sum_{n=0}^{\infty} \frac{1}{(n+1)}.\end{aligned}$$

This series is just the harmonic series, so it diverges. Hence the series diverges at $x = -3$ and therefore -3 *is not* included on the interval of convergence.

Now do the same for the other value of x. Plugging $x = 5$:

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{(-1)^n}{4^n(n+1)}(5-1)^n &= \sum_{n=0}^{\infty} \frac{(-1)^n}{4^n(n+1)}4^n \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)}.\end{aligned}$$

This series converges by the alternating series test. Hence $x = 5$ *is* included on the interval of convergence.

We conclude therefore that the interval of convergence is $I = (-3, 5]$.

3. Maclaurin series.

The basic formula for the Maclaurin series is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n, \tag{1}$$

where $f^{(n)}(0)$ denotes the n^{th} derivative of f *evaluated at* 0 — i.e., take n derivatives and then plug in $x = 0$. Notice that $f^{(0)}$, i.e., $f^{(n)}$ with $n = 0$, simply means the function itself: $f^{(0)}(x) = f(x)$. To find the Maclaurin series, use the following steps, which we will illustrate with the example $f(x) = \frac{7^x}{3}$.

STEP 1. Take a few derivatives of $f(x)$, plug in $x = 0$ and figure out a general formula for $f^{(n)}(0)$.

$$
\begin{aligned}
&n = 0,\ f^{(0)}(x) = f(x) = \frac{7^x}{3} \Rightarrow f(0) = \frac{1}{3},\\
&n = 1,\ f'(x) = \frac{7^x}{3}\ln 7 \Rightarrow f'(0) = \frac{\ln 7}{3},\\
&n = 2,\ f''(x) = \frac{7^x}{3}(\ln 7)^2 \Rightarrow f''(0) = \frac{(\ln 7)^2}{3},\\
&n = 3,\ f'''(x) = \frac{7^x}{3}(\ln 7)^3 \Rightarrow f'''(0) = \frac{(\ln 7)^3}{3},\\
&n = 4,\ f''''(x) = \frac{7^x}{3}(\ln 7)^4 \Rightarrow f''''(0) = \frac{(\ln 7)^4}{3},
\end{aligned}
$$

we see that there is a clear patter, so the n^{th} derivative evaluated at zero is given by

$$f^{(n)}(0) = \frac{(\ln 7)^n}{3}.$$

STEP 2. Plug the formula you found for $f^{(n)}(0)$ into (1):

$$\frac{7^x}{3} = \sum_{n=0}^{\infty} \frac{(\ln 7)^n}{3n!} x^n. \tag{2}$$

A **common mistake** is the following: after computing the derivatives, students forget to plug in $x = 0$. Then they use the formula for $f^{(n)}(x)$ into (1); in our example this would lead to

$$\frac{7^x}{3} = \sum_{n=0}^{\infty} \frac{7^x(\ln 7)^n}{3n!} x^n.$$

This formula is *wrong*. One way of keeping track of what you are doing is to remember that *the only way x can appear in the Maclaurin series is raised to a power of n*; if you have x appearing in any other way (7^x, $\cos x$ etc), then there is a mistake.

STEP 3. If asked, find the radius and interval of convergence of the series. Notice that the Maclaurin series found in step 2 (formula (2) in our example), is just a power series, so to find its radius and interval of convergence, use the steps given in sections 1 and 2 above.

Important: make sure to read carefully the question: many problems with Maclaurin and Taylor series ask you to find the radius of convergence, but *don't* ask the *interval* of convergence. Still, many times students spend time during the exams finding the interval of convergence, even if not asked.

4. Common Maclaurin series.

You should memorize the following Maclaurin series, along with their radius of convergence.

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \;\; R = 1 \text{ (geometric series)}$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \;\; R = \infty$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}, \;\; R = \infty$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}, \;\; R = \infty$$

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}, \;\; R = 1$$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}, \;\; R = 1$$

Here are some hints that can help you to memorize or derive these formulas if needed.

- Try to remember the geometric series. This shouldn't be very hard since we have been working with it for a while now.
- The series for e^x can be quickly derived from formula (1): since the derivative of e^x is always itself, for $f(x) = e^x$ we have $f^{(n)}(0) = 1$, then just plug this into (1) (the $n!$ comes from the formula itself).
- For $\cos x$ and $\sin x$, use the formula for e^x but replace n by $2n$ in the case of $\cos x$ and n by $2n+1$ in the case of $\sin x$, and in both cases make the series alternating by putting a $(-1)^n$ in front. If you get confused about using $2n$ or $2n+1$, remember that $\cos x$ is an even function, so the powers of x for $\cos x$ have to be even, hence $2n$, whereas $\sin x$ is an odd function, so the powers of x for $\sin x$ have to be odd, hence $2n+1$.
- For $\arctan x$ and $\ln(1+x)$, you can get the power series from the geometric series by integration, since

  $$\arctan x = \int \frac{1}{1+x^2}\, dx,$$

  and

  $$\ln(1+x) = \int \frac{1}{1+x}\, dx.$$

  (when you do these integrals there will be a constant of integration, but as we saw in class the constant will be zero, so just ignore it).
- To memorize the radius of convergence of such functions, remember that the radius of convergence of the geometric series is $R = 1$. Therefore the radius of convergence of those Maclaurin series obtained from geometric series, namely, $\arctan x$ and $\ln(1+x)$, will also be $R = 1$. The remaining ones $(e^x, \sin x, \cos x)$ will have $R = \infty$.

5. Taylor series.

The basic formula for the Taylor series centered at a is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n, \tag{3}$$

where $f^{(n)}(a)$ denotes the n^{th} derivative of f *evaluated at* a — i.e., take n derivatives and then plug in $x = a$. To find the Taylor series, proceed exactly as in the Maclaurin series, but instead of plugging $x = 0$ in the derivatives, plug in $x = a$.

Example. Find the Taylor series for $f(x) = \frac{7^x}{3}$ centered at $x = 3$.

Proceeding as in the example of the Maclaurin series we find

$$f^{(n)}(3) = \frac{7^3(\ln 7)^n}{3}.$$

Then, plug this into (3) to find

$$\frac{7^x}{3} = \sum_{n=0}^{\infty} \frac{7^3(\ln 7)^n}{3n!}(x-3)^n.$$

Remark. The Maclaurin series is simply the Taylor series when $a = 0$.