MAT 155B - FALL 12 - SUMMARY OF CONVERGENCE TESTS

All references to pages and examples are from the textbook.

Important Series.

Geometric series:

$$\sum_{n=0}^{\infty} ar^n = \begin{cases} \frac{a}{1-r} & \text{if } |r| < 1, \\ \text{divergent} & \text{if } |r| \geq 1. \end{cases}$$

See example 4, p. 731.

Harmonic series:

$$\sum_{n=1}^{\infty} \frac{1}{n} = \infty \text{ (divergent)}$$

p-series:

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \begin{cases} \text{convergent} & \text{if } p > 1, \\ \text{divergent} & \text{if } p \leq 1. \end{cases}$$

Alternating series: a series where the terms alternative between positive and negative terms,

$$\sum_{n=1}^{\infty} (-1)^n b_n \text{ where } b_n > 0.$$

Notice that $(-1)^n = 1$ if n is even and $(-1)^n = -1$ if n is odd.

Telescoping series: a series where the n^{th} term cancels with the $(n+1)^{th}$ term, e.g.

$$\sum_{n=1}^{\infty} = \frac{1}{n(n+1)} = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{3} + \frac{1}{4} - \dots$$

See example 7, p. 731.

Convergence tests for series.

Divergence test: if $\lim_{n\to\infty} a_n$ does not exist or $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ diverges. VERY COMMON MISTAKE: if $\lim_{n\to\infty} a_n = 0$, you *cannot* conclude that the series converges. For example, $\lim_{n\to\infty} \frac{1}{n} = 0$ but the harmonic series diverges.

Integral test: $f(x)$ a function which is continuous, positive and decreasing on for $x \geq 1$. $a_n = f(n)$. Then

$$(a) \text{ if } \int_1^{\infty} f(x)dx \text{ converges, then } \sum_{n=1}^{\infty} a_n \text{ converges.}$$

$$(b) \text{ if } \int_1^{\infty} f(x)dx \text{ diverges, then } \sum_{n=1}^{\infty} a_n \text{ diverges.}$$

See example 4 p. 741.

Comparison test: a_n and b_n *positive* terms.

$$(a) \text{ if } \sum_{n=1}^{\infty} b_n \text{ converges and } a_n \leq b_n \text{ then } \sum_{n=1}^{\infty} a_n \text{ converges.}$$

$$(b) \text{ if } \sum_{n=1}^{\infty} b_n \text{ diverges and } a_n \geq b_n \text{ then } \sum_{n=1}^{\infty} a_n \text{ diverges.}$$

See examples 2 p. 747. COMMON MISTAKE: $\sum_{n=1}^{\infty} b_n$ diverges and $a_n \leq b_n$ then you *cannot* conclude that $\sum_{n=1}^{\infty} a_n$ diverges; if $\sum_{n=1}^{\infty} b_n$ converges and $a_n \geq b_n$ then you *cannot* conclude that $\sum_{n=1}^{\infty} a_n$ converges.

Limit comparison test: a_n and b_n *positive* terms. If

$$\lim_{n\to 0} \frac{a_n}{b_n} = c$$

where c is not ∞ neither zero, then either both series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge or both diverge. See example 4 p. 749.

Alternating series test: if the series

$$\sum_{n=1}^{\infty} (-1)^n b_n \text{ where } \; b_n > 0.$$

satisfies (i) $b_{n+1} \leq b_n$ for all n, and (ii) $b_n \to 0$ as $n \to \infty$, then the series converges.

Absolute convergence: A series $\sum_{n=1}^{\infty} a_n$ is absolutely convergence if $\sum_{n=1}^{\infty} |a_n|$ converges. If $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n|$ diverges, the series is called conditionally convergent. See example 3 p. 757. Every absolutely convergent series is also convergent; the reciprocal is *not* true.

Ratio test: Compute

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L.$$

If $L < 1$ then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, and hence convergent; if $L > 1$ then the series $\sum_{n=1}^{\infty} a_n$ is divergent. If $L = 1$, no conclusion can be drawn about the convergence/divergence of $\sum_{n=1}^{\infty} a_n$. See examples 4 and 5 p. 759.

Root test: Compute

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = L.$$

If $L < 1$ then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, and hence convergent; if $L > 1$ then the series $\sum_{n=1}^{\infty} a_n$ is divergent. If $L = 1$, no conclusion can be drawn about the convergence/divergence of $\sum_{n=1}^{\infty} a_n$. See example 6 p. 760.

URL: `http://www.disconzi.net/Teaching/MAT155B-Fall-12/MAT155B-Fall-12.html`