Question 1 [20 pts]. Find the radius and interval of convergence of the series.

(a) [5 pts]. $\sum_{n=1}^{\infty} \frac{x^n}{5^n}$

$$\sum_{n=1}^{\infty} \frac{x^n}{5^n} = \sum_{n=1}^{\infty} \left(\frac{x}{5}\right)^n, \quad \left|\frac{x}{5}\right| < 1 \quad \text{so} \quad |x| < 5$$

radius of convergence is $R = 5$

- $\sum_{n=1}^{\infty} \frac{5^n}{5^n} = \sum_{n=1}^{\infty} 1$ diverges
- $\sum_{n=1}^{\infty} \frac{(-5)^n}{5^n} = \sum_{n=1}^{\infty} (-1)^n$ diverges, interval of convergence is $(-5,5)$.

(b) [5 pts]. $\sum_{n=1}^{\infty} \frac{n}{2^n}(x-1)^n$

$$\lim_{n\to\infty} \left| \frac{\left(\frac{(n+1)(x-1)^{n+1}}{2^{n+1}}\right)}{\left(\frac{n(x-1)^n}{2^n}\right)} \right| = \lim_{n\to\infty} \left| \left(\frac{n+1}{n}\right) \frac{x-1}{2} \right| = \left| \frac{x-1}{2} \right| < 1$$

$$|x-1| < 2$$

radius of convergence is 2

- $\sum_{n=1}^{\infty} n \frac{(-2)^n}{2^n} = \sum_{n=1}^{\infty} n(-1)^n$ diverges
- $\sum_{n=1}^{\infty} n\left(\frac{2^n}{2^n}\right) = \sum_{n=1}^{\infty} n$ diverges, interval of convergence is $(-1,3)$.

(c) [5 pts]. $\sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n}}$

$$\lim_{n\to\infty} \left| \frac{\left(\frac{x^{n+1}}{\sqrt{n+1}}\right)}{\left(\frac{x^n}{\sqrt{n}}\right)} \right| = \lim_{n\to\infty} \left| \frac{x}{\sqrt{\frac{n+1}{n}}} \right| = |x| < 1$$

radius of convergence is $R=1$

- $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converges
- $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges, interval of convergence is $[-1, 1)$.

(d) [5 pts]. $\sum_{n=0}^{\infty} \frac{n^{\frac{1}{3}}}{(2n+1)!}(x+3)^n$

$$\lim_{n\to\infty} \left| \frac{\left(\frac{(n+1)^{1/3}(x+3)^{n+1}}{(2n+3)!}\right)}{\left(\frac{n^{1/3}(x+3)^n}{(2n+1)!}\right)} \right| = \lim_{n\to\infty} \left| \frac{\left(\frac{n+1}{n}\right)^{1/3}(x+3)}{(2n+3)(2n+2)} \right| = 0 < 1$$

radius of convergence is $R = \infty$

interval of convergence is $(-\infty, \infty)$.

Question 2 [20 pts]. Find the Maclaurin series for $f(x)$ and its radius of convergence.

(a) [4 pts]. $f(x) = \cos^2 x$

$$f(x) = \frac{1+\cos 2x}{2}$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}, \quad \cos 2x = \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n}}{(2n)!}$$

$$1+\cos 2x = 2 + \sum_{n=1}^{\infty} \frac{(-1)^n 2^{2n} x^{2n}}{(2n)!}$$

$$f(x) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n 2^{2n-1} x^{2n}}{(2n)!}, \quad \text{radius } R = \infty$$

(b) [4 pts]. $f(x) = x^2 \arctan \frac{x}{3}$

$$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$

$$\tan^{-1}\left(\frac{x}{3}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{x}{3}\right)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{3^{2n+1}(2n+1)}$$

$$F(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+3}}{3^{2n+1}(2n+1)}, \quad \text{radius } R = 3$$

(c) [4 pts]. $f(x) = e^{3x}$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$e^{3x} = \sum_{n=0}^{\infty} \frac{(3x)^n}{n!} = \sum_{n=0}^{\infty} \frac{3^n x^n}{n!}, \text{ radius } R = \infty$$

(d) [4 pts]. $f(x) = \dfrac{1}{(3+x)^2}$

$$f(x) = -\frac{d}{dx}\left(\frac{1}{3+x}\right)$$

$$\frac{1}{3+x} = \frac{1}{3}\left(\frac{1}{1+\frac{x}{3}}\right) = \frac{1}{3}\left(\frac{1}{1-(-\frac{x}{3})}\right) = \frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{-x}{3}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{3^{n+1}}$$

$$f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} n \, x^{n-1}}{3^{n+1}}, \text{ radius } R = 3$$

(e) [4 pts]. $f(x) = \ln(1 + 2\sqrt{x} + x)$

Since the domain of f is $[0, \infty)$, the radius of convergence can only be $R = 0$.

Thus, $f(x) = 0$ is the Maclaurin series.

Question 3 [15 pts]. Find the Taylor series for $f(x)$ centered at the given value of a. You do not have to determine its radius or interval of convergence.

(a) [10 pts]. $f(x) = \frac{1}{x^3}$, $a = 2$

$$f'(x) = \frac{-3}{x^4}$$

$$f''(x) = \frac{12}{x^5}$$

$$f'''(x) = \frac{-60}{x^6}$$

$$\vdots$$

$$f^{(n)}(x) = \frac{(-1)^n (n+2)!}{2x^{n+3}}$$

$$f^{(n)}(2) = \frac{(-1)^n (n+2)!}{2^{n+4}}$$

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (n+2)!}{2^{n+4}\, n!} (x-2)^n$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (n+2)(n+1)}{2^{n+4}} (x-2)^n$$

(b) [5 pts]. $f(x) = \sin x$, $a = -\pi$

$$f'(x) = \cos x \qquad f(-\pi) = 0$$

$$f''(x) = -\sin x \qquad f'(-\pi) = -1$$

$$f'''(x) = -\cos x \qquad f''(-\pi) = 0$$

$$f^{(4)}(x) = \sin x \qquad f'''(-\pi) = 1$$

$$\vdots \qquad f^{(4)}(-\pi) = 0$$

$$\vdots$$

$$\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n+1)!} (x+\pi)^{2n+1}$$

Question 4 [10 pts]. Evaluate the integral as an infinite series.

(a) [5 pts]. $\int e^{-x^2}\,dx$

$$= \int \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!}\,dx$$

$$= \int \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}\,dx$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)\,n!} + C$$

(b) [5 pts]. $\int \frac{\ln(1+x^4)}{x}\,dx$

$$= \int \frac{\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x^4)^n}{n}}{x}\,dx$$

$$= \int \frac{\sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{4n}}{n}}{x}\,dx$$

$$= \int \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{4n-1}}{n}\,dx$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{4n}}{4n^2} + C$$

Question 5 [10 pts]. Evaluate the given definite integral

$$\int_0^1 \arctan(x^6)\,dx$$

as a power series, and determine how many terms of the resulting series have to be used in order to approximate the integral to within two decimal places.

$$\int_0^1 \arctan(x^6)\,dx = \int_0^1 \sum_{n=0}^{\infty} \frac{(-1)^n (x^6)^{2n+1}}{2n+1}\,dx$$

$$= \int_0^1 \sum_{n=0}^{\infty} \frac{(-1)^n x^{12n+6}}{2n+1}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{12n+7}}{(12n+7)(2n+1)}\Bigg|_0^1$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(12n+7)(2n+1)}$$

When $n = 2$, $\frac{1}{(31)(5)} < \frac{1}{100}$, so 2 terms will suffice.

Question 6 [10 pts]. Find the sum of the series.

(a) [5 pts]. $\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!}$

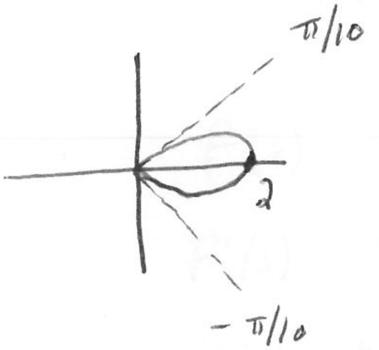
$$e^{-1/2} = \sum_{n=0}^{\infty} \frac{(-1/2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!}$$

(b) [5 pts]. $\sum_{n=0}^{\infty} \frac{(-1)^n e^{2n+1}}{(2n+1)!}$

$$\sin(e) = \sum_{n=0}^{\infty} \frac{(-1)^n e^{2n+1}}{(2n+1)!}$$

Question 7 [10 pts]. Write an integral representing the area enclosed by one loop of the given curve (do not evaluate the integral)

$$r = 2\cos(5\theta).$$



$$\int_{-\pi/10}^{\pi/10} \frac{1}{2}\left(2\cos(5\theta)\right)^2 d\theta$$

Question 8 [5 pts]. Write an integral representing the length of the given curve (do not evaluate the integral)

$$r = \ln\theta,\ \frac{\pi}{2} \le \theta \le \pi.$$

$$\frac{dr}{d\theta} = \frac{1}{\theta}$$

$$\int_{\pi/2}^{\pi} \sqrt{(\ln\theta)^2 + \frac{1}{\theta^2}}\, d\theta$$

Extra credit [5 pts]. Let k be any real number. Show that

$$(1+x)^k = \sum_{n=0}^{\infty} \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!} x^n,$$

with radius of convergence $R = 1$.

$f(x) = (1+x)^k$

$f'(x) = k(1+x)^{k-1}$

$f''(x) = k(k-1)(1+x)^{k-2}$

$\vdots$

For $k=0$, both sides equal 1.

$n \geq 1$, $f^{(n)}(x) = k(k-1)\cdots(k-n+1)(1+x)^{k-n}$

$f(0) = 1$

$f^{(n)}(0) = k(k-1)\cdots(k-n+1)$

$$(1+x)^k = \sum_{n=0}^{\infty} \frac{k(k-1)\cdots(k-n+1)}{n!} x^n$$

$$\lim_{n\to\infty} \left| \frac{\left(\frac{k(k-1)\cdots(k-n+1)(k-n)}{(n+1)!} x^{n+1}\right)}{\left(\frac{k(k-1)\cdots(k-n+1)}{n!}\right) x^n} \right| = \lim_{n\to\infty} \left| \frac{k-n}{n+1} x \right| = |x| < 1$$