VANDERBILT UNIVERSITY
MAT 155B, FALL 12 — SOLUTIONS TO TEST 2

INSTRUCTIONS: In questions 1 to 4 below, compute the given integrals.

Question 1 [10 pts]. $\int e^{\sqrt[3]{x}}\, dx$

Put

$$z = \sqrt[3]{x} \Rightarrow dz = \frac{1}{3}x^{-\frac{2}{3}}\, dx \Rightarrow dx = 3x^{\frac{2}{3}}\, dz = 3z^2\, dz,$$

where in the last step we used $x = z^3$. Then

$$\int e^{\sqrt[3]{x}}\, dx = 3\int z^2 e^z\, dz.$$

Integrate by parts with

$$u = z^2 \Rightarrow du = 2z\, dz,$$
$$dv = e^z\, dz \Rightarrow v = e^z,$$

to get

$$3\int z^2 e^z\, dz = 3\Big(z^2 e^z - 2\int z e^z\, dz\Big) = 3z^2 e^z - 6\int z e^z\, dz.$$

Integrate by parts again with

$$u = z \Rightarrow du = dz,$$
$$dv = e^z\, dz \Rightarrow v = e^z,$$

so that

$$3z^2 e^z - 6\int z e^z\, dz = 3z^2 e^z - 6\Big(z e^z - \int e^z\, dz\Big) = 3z^2 e^z - 6z e^z + 6e^z + C.$$

Plugging back $z = \sqrt[3]{x}$ yields

$$\int e^{\sqrt[3]{x}}\, dx = 3x^{\frac{2}{3}} e^{\sqrt[3]{x}} - 6\sqrt[3]{x} e^{\sqrt[3]{x}} + 6e^{\sqrt[3]{x}} + C.$$

Question 2 [10 pts]. $\int \tan^5\theta\, \sec^3\theta\, d\theta$

Recall that $\tan^2\theta + 1 = \sec^2\theta$ and that $(\sec\theta)' = \sec\theta\tan\theta$. Then

$$\begin{aligned}\int \tan^5\theta\, \sec^3\theta\, d\theta &= \int \tan^4\theta \sec^2\theta \tan\theta \sec\theta\, d\theta \\ &= \int (\sec^2\theta - 1)^2 \sec^2\theta \tan\theta \sec\theta\, d\theta.\end{aligned}$$

Putting $u = \sec\theta$,

$$\begin{aligned}\int (\sec^2\theta - 1)^2 \sec^2\theta \tan\theta \sec\theta \, d\theta &= \int (u^2-1)^2 u^2 \, du = \int (u^4 - 2u^2 + 1) u^2 \, du \\ &= \int (u^6 - 2u^4 + u^2)\, du = \frac{1}{7}u^7 - \frac{2}{5}u^5 + \frac{1}{3}u^3 + C \\ &= \frac{1}{7}(\sec\theta)^7 - \frac{2}{5}(\sec\theta)^5 + \frac{1}{3}(\sec\theta)^3 + C.\end{aligned}$$

Question 3 [10 pts]. $\displaystyle\int \frac{x^2}{(4-x^2)^{\frac{3}{2}}}\, dx$

Put $x = 2\sin\theta$, so $dx = 2\cos\theta\, d\theta$ and

$$\begin{aligned}\int \frac{x^2}{(4-x^2)^{\frac{3}{2}}}\, dx &= \int \frac{4\sin^2\theta}{(4-4\sin^2\theta)^{\frac{3}{2}}} 2\cos\theta\, d\theta = \frac{8}{4^{\frac{3}{2}}} \int \frac{\sin^2\theta}{\cos^2\theta}\, d\theta \\ &= \frac{8}{8}\int \frac{1-\cos^2\theta}{\cos^2\theta}\, d\theta = \int \sec^2\theta\, d\theta - \int d\theta \\ &= \tan\theta - \theta + C.\end{aligned}$$

But $\theta = \arcsin\frac{x}{2}$ and $\tan\theta = \frac{x}{\sqrt{4-x^2}}$, so

$$\int \frac{x^2}{(4-x^2)^{\frac{3}{2}}}\, dx = \frac{x}{\sqrt{4-x^2}} - \arcsin\frac{x}{2} + C.$$

Question 4 [10 pts]. $\displaystyle\int x\cos^2 x\, dx$

Recalling that $\cos^2 x = \frac{1+\cos(2x)}{2}$,

$$\int x\cos^2 x\, dx = \int x\frac{1+\cos(2x)}{2}\, dx = \frac{1}{4}x^2 + \frac{1}{2}\int x\cos(2x)\, dx.$$

Integrate by parts with

$$u = x \Rightarrow du = dx,$$

$$dv = \cos(2x)\, dx \Rightarrow v = \frac{\sin(2x)}{2},$$

to get

$$\begin{aligned}\int x\cos^2 x\, dx &= \frac{1}{4}x^2 + \frac{1}{2}\int x\cos(2x)\, dx \\ &= \frac{1}{4}x^2 + \frac{1}{2}\left(x\frac{\sin(2x)}{2} - \int \frac{\sin(2x)}{2}\, dx\right) \\ &= \frac{1}{4}x^2 + \frac{1}{4}x\sin(2x) + \frac{1}{8}\cos(2x) + C.\end{aligned}$$

INSTRUCTIONS: In questions 5 to 7 below, write out the form of the partial fraction decomposition of the function. *Do not* determine the numerical values of the coefficients. For example, given:

$$\frac{x+7}{x^3+4x^2+4x},$$

write

$$\frac{x+7}{x^3+4x^2+4x} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{(x+2)^2},$$

and that is the answer, i.e., you do not have to find the values of A, B and C.

Question 5 [5 pts]. $\dfrac{3x^3-x^2+6x-4}{(x^2+1)(x^4+4x^2+4)}$

$$\begin{aligned}\frac{3x^3-x^2+6x-4}{(x^2+1)(x^4+4x^2+4)} &= \frac{3x^3-x^2+6x-4}{(x^2+1)(x^2+2)^2}\\ &= \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+2} + \frac{Ex+F}{(x^2+2)^2}.\end{aligned}$$

Question 6 [5 pts]. $\dfrac{x^5+x-1}{x^3+1}$

Use long division to find

$$\frac{x^5+x-1}{x^3+1} = x^2 + \frac{-x^2+x-1}{x^3+1},$$

so that

$$\begin{aligned}\frac{x^5+x-1}{x^3+1} &= x^2 + \frac{-x^2+x-1}{x^3+1}\\ &= x^2 + \frac{-x^2+x-1}{(x+1)(x^2-x+1)}\\ &= x^2 + \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1}.\end{aligned}$$

Question 7 [10 pts]. Find the partial fraction decomposition of $\frac{1}{x^2-2x}$ (notice that here you *do have* to find the values of the constants).

Write

$$\begin{aligned}\frac{1}{x^2-2x} = \frac{1}{x(x-2)} &= \frac{A}{x} + \frac{B}{x-2}\\ &= \frac{A(x-2)+Bx}{x(x-2)}.\end{aligned}$$

Hence,

$$A(x-2)+Bx = 1.$$

Plugging $x=2$ we find $B=\frac{1}{2}$; plugging $x=0$ we find $A=-\frac{1}{2}$.

INSTRUCTIONS: In questions 8 to 10, determine whether the given integral is convergent.

Question 8 [10 pts]. $\displaystyle\int_1^\infty \frac{2+\cos x}{x^4}\,dx$

Since $-1 \leq \cos x \leq 1$, we have that $2+\cos x$ is is always positive and satisfies $2+\cos x \leq 3$. Therefore

$$\int_1^\infty \frac{2+\cos x}{x^4}\,dx \leq 3\int_1^\infty \frac{1}{x^4}\,dx.$$

The latter is a p-integral with $p > 1$ (i.e, $p = 4$), so it converges. We conclude that $\int_1^\infty \frac{2+\cos x}{x^4}\,dx$ converges as well.

Question 9 [10 pts]. $\displaystyle\int_2^\infty \frac{x+1}{\sqrt{x^4-x}}\,dx$

For any x on the domain of integration:

$$\frac{1}{x^4-x} \geq \frac{1}{x^4},$$

so that

$$\begin{aligned}\int_2^\infty \frac{x+1}{\sqrt{x^4-x}} &\geq \int_2^\infty \frac{x+1}{\sqrt{x^4}}\,dx \\ &= \int_2^\infty \frac{1}{x}\,dx + \int_2^\infty \frac{1}{x^2}\,dx.\end{aligned}$$

But $\int_2^\infty \frac{1}{x}\,dx$ is a p-integral with $p \leq 1$ (i.e, $p = 1$), so it diverges. Hence $\int_2^\infty \frac{x+1}{\sqrt{x^4-x}}\,dx$ diverges as well.

Question 10 [10 pts]. $\displaystyle\int_0^1 \frac{\ln x}{\sqrt{x}}\,dx$

Put

$$\int_0^1 \frac{\ln x}{\sqrt{x}}\,dx = \lim_{t\to 0^+}\int_t^1 \frac{\ln x}{\sqrt{x}}\,dx,$$

and integrate by parts with

$$\begin{aligned}u = \ln x &\Rightarrow du = \frac{1}{x}\,dx,\\ dv = \frac{1}{\sqrt{x}}\,dx &\Rightarrow v = 2\sqrt{x},\end{aligned}$$

to get

$$\begin{aligned}\int_t^1 \frac{\ln x}{\sqrt{x}}\,dx &= 2\sqrt{x}\ln x\Big|_t^1 - 2\int_t^1 \frac{1}{\sqrt{x}}\,dx \\ &= 2\sqrt{x}\ln x\Big|_t^1 - 4\sqrt{x}\Big|_t^1 = -2\sqrt{t}\ln t - 4 + 4\sqrt{t}.\end{aligned}$$

Since $\sqrt{t}\ln t \to 0$ as $t \to 0^+$ (use L'Hospital), we conclude that the integral is convergent.

Question 11 [10 pts]. Set up an integral for the area of the surface obtained by rotating $y = e^{x^4}$, $4 \leq x \leq 6$, about the x-axis. Do not evaluate the integral.

The formula is

$$S = \int_a^b 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx.$$

Since $y' = 4x^3e^{x^4}$, we obtain

$$S = \int_4^6 2\pi e^{x^4}\sqrt{1+(4x^3e^{x^4})^2}\,dx.$$

Extra Credit [10 pts]. Let f be a non-negative continuous functions defined on $[0,1]$. Let y satisfy

$$y''(x) - f(x)y(x) = 0 \quad \text{on} \quad (0,1).$$

Assume further that $y(0) = y(1) = 0$. Show that $y(x) = 0$ for all $x \in [0,1]$ (*hint:* integration by parts is useful here).

Multiply the equation by $y(x)$ and integrate between 0 and 1 to get

$$\int_0^1 y(x)y''(x)\,dx - \int_0^1 f(x)(y(x))^2\,dx = 0.$$

Integrate by parts the first integral with

$$u = y(x) \Rightarrow du = y'(x)\,dx,$$
$$dv = y''(x)\,dx \Rightarrow v = y'(x),$$

to obtain

$$y(x)y'(x)\Big|_0^1 - \int_0^1 (y'(x))^2\,dx - \int_0^1 f(x)(y(x))^2\,dx = 0.$$

But

$$y(x)y'(x)\Big|_0^1 = y(1)y'(1) - y(0)y'(0) = 0,$$

since $y(1) = y(0) = 0$ by hypothesis. Therefore

$$-\int_0^1 (y'(x))^2\,dx - \int_0^1 f(x)(y(x))^2\,dx = 0,$$

or equivalently,

$$\int_0^1 (y'(x))^2\,dx + \int_0^1 f(x)(y(x))^2\,dx = 0. \tag{1}$$

Since $f(x) \geq 0$, it follows that $f(x)(y(x))^2 \geq 0$, from which we conclude

$$\int_0^1 f(x)(y(x))^2\,dx \geq 0. \tag{2}$$

Also,

$$\int_0^1 (y'(x))^2\,dx \geq 0. \tag{3}$$

But the only way that a sum of two non-negative numbers can equal zero, is if each number vanishes separately. Hence from (1), (2), and (3) we conclude that

$$\int_0^1 (y'(x))^2\,dx = 0.$$

But the integral of a non-negative (continuous) function can be zero only if the function itself equals zero, so

$$(y'(x))^2 = 0, \quad \text{which gives} \quad y'(x) = 0.$$

From this we conclude that $y(x)$ is constant. But since $y(0) = 0$, this constant is necessarily zero.