MAT 155B - FALL 12 - SOLUTIONS TO PRACTICE TEST 2

INSTRUCTIONS: In questions 1 to 6 below, compute the given integrals.

Question 1. $\displaystyle\int \frac{\cos^5\theta}{\sqrt{\sin\theta}}\,d\theta.$

Write

$$\int \frac{\cos^5\theta}{\sqrt{\sin\theta}}\,d\theta = \int \frac{\cos^4\theta\cos\theta}{\sqrt{\sin\theta}}\,d\theta = \int \frac{(1-\sin^2\theta)^2\cos\theta}{\sqrt{\sin\theta}}\,d\theta.$$

Put $u = \sin\theta$:

$$\begin{aligned}\int \frac{(1-\sin^2\theta)^2\cos\theta}{\sqrt{\sin\theta}}\,d\theta = \int \frac{(1-u^2)^2}{\sqrt{u}}\,du &= \int (u^{-\frac{1}{2}} - 2u^{\frac{3}{2}} + u^{\frac{7}{2}})\,du \\ &= 2\sqrt{u} - \frac{4}{5}u^{\frac{5}{2}} + \frac{2}{9}u^{\frac{9}{2}} + C \\ &= 2\sqrt{\sin\theta} - \frac{4}{5}(\sin\theta)^{\frac{5}{2}} + \frac{2}{9}(\sin\theta)^{\frac{9}{2}} + C\end{aligned}$$

Question 2. $\displaystyle\int e^{x+e^x}\,dx.$

Write $e^{x+e^x} = e^x\,e^{e^x}$ and put $u = e^x$:

$$\int e^{x+e^x}\,dx = \int e^x e^{e^x}\,dx = \int e^u\,du = e^u + C = e^{e^x} + C.$$

Question 3. $\displaystyle\int x^{-3}(\ln x)^2\,dx.$

Integrate by parts:

$$\begin{aligned}u = (\ln x)^2 &\Rightarrow du = 2\ln x\frac{1}{x}\,dx \\ dv = \frac{1}{x^3}\,dx &\Rightarrow v = -\frac{1}{2x^2},\end{aligned}$$

$$\int x^{-3}(\ln x)^2\,dx = -\frac{(\ln x)^2}{2x^2} + \int \frac{\ln x}{x^3}\,dx.$$

Integrate by parts again:

$$\begin{aligned}u = \ln x &\Rightarrow du = \frac{1}{x}\,dx \\ dv = \frac{1}{x^3}\,dx &\Rightarrow v = -\frac{1}{2x^2},\end{aligned}$$

$$\int x^{-3}(\ln x)^2\,dx = -\frac{(\ln x)^2}{2x^2} + \int \frac{\ln x}{x^3}\,dx.$$
$$= -\frac{(\ln x)^2}{2x^2} - \frac{\ln x}{2x^2} - \frac{1}{4x^2} + C.$$

Question 4. $\displaystyle\int_1^2 \frac{\ln(\ln x)}{x}\,dx.$

Put $u = \ln x$:

$$\int_1^2 \frac{\ln(\ln x)}{x}\,dx = \int_0^{\ln 2} \ln u\,du = \Big[u\ln u - u\Big]_0^{\ln 2} = \ln 2\ln(\ln 2) - \ln 2,$$

where we used L'Hospital to compute $\lim_{u\to 0^+} u\ln u = 0$.

Question 5. $\displaystyle\int_0^3 x^2\sqrt{9-x^2}\,dx.$

Put $x = 3\sin\theta$:

$$\int_0^3 x^2\sqrt{9-x^2}\,dx = 81\int_0^{\frac{\pi}{2}} \sin^2\theta\cos^2\theta\,d\theta = 81\int_0^{\frac{\pi}{2}} (\sin\theta\cos\theta)^2\,d\theta$$
$$= \frac{81}{4}\int_0^{\frac{\pi}{2}} (\sin(2\theta))^2\,d\theta,$$

where we used $\sin(2\theta) = 2\sin\theta\cos\theta$; continuing:

$$\int_0^3 x^2\sqrt{9-x^2}\,dx = \frac{81}{4}\int_0^{\frac{\pi}{2}} (\sin(2\theta))^2\,d\theta = \frac{81}{4}\int_0^{\frac{\pi}{2}} \frac{1-\cos(4\theta)}{2}\,d\theta$$
$$= \frac{81}{8}\theta\Big|_0^{\frac{\pi}{2}} - \frac{81}{8}\frac{\sin(4\theta)}{4}\Big|_0^{\frac{\pi}{2}}$$
$$= \frac{81\pi}{16},$$

where we used the identity $\sin^2\alpha = \frac{1-\cos(2\alpha)}{2}$.

Question 6. $\displaystyle\int_0^{\frac{1}{2}} \frac{xe^{2x}}{(1+2x)^2}\,dx.$

Put $z = 1+2x$, so $x = \frac{z-1}{2}$ and

$$\int_0^{\frac{1}{2}} \frac{xe^{2x}}{(1+2x)^2}\,dx = \frac{1}{4e}\int_1^2 \frac{z-1}{z^2}e^z\,dz = \frac{1}{4e}\int_1^2 \frac{e^z}{z}\,dz - \frac{1}{4e}\int_1^2 \frac{e^z}{z^2}\,dz.$$

Integrate by parts the second integral with

$$u = e^z \Rightarrow du = e^z\,dz$$
$$dv = \frac{1}{z^2}\,dz \Rightarrow v = -\frac{1}{z},$$

so

$$\begin{aligned}\int_0^{\frac{1}{2}} \frac{xe^{2x}}{(1+2x)^2}\,dx &= \frac{1}{4e}\int_1^2 \frac{e^z}{z}\,dz - \frac{1}{4e}\int_1^2 \frac{e^z}{z^2}\,dz \\ &= \frac{1}{4e}\int_1^2 \frac{e^z}{z}\,dz + \frac{1}{4e}\frac{e^z}{z}\Big|_1^2 - \frac{1}{4e}\int_1^2 \frac{e^z}{z}\,dz \\ &= \frac{1}{4e}\frac{e^z}{z}\Big|_1^2 = \frac{1}{8}(e-2).\end{aligned}$$

INSTRUCTIONS: In questions 7 to 10 below, write out the form of the partial fraction decomposition of the function. *Do not* determine the numerical values of the coefficients. For example, given:

$$\frac{x+7}{x^3+4x^2+4x},$$

write

$$\frac{x+7}{x^3+4x^2+4x} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{(x+2)^2},$$

and that is the answer, i.e., you do not have to find the values of A, B and C.

Question 7. $\dfrac{x}{(x+4)(2x-1)}$.

$$\frac{x}{(x+4)(2x-1)} = \frac{A}{x+4} + \frac{B}{2x-1}.$$

Question 8. $\dfrac{x^3-4x-10}{x^2-x-6}$.

First use long division to write

$$\frac{x^3-4x-10}{x^2-x-6} = x+1+\frac{3x-4}{x^2-x-6}.$$

Then

$$\frac{3x-4}{x^2-x-6} = \frac{3x-4}{(x-3)(x+2)} = \frac{A}{x-3} + \frac{B}{x+2},$$

so

$$\frac{x^3-4x-10}{x^2-x-6} = x+1+\frac{A}{x-3} + \frac{B}{x+2}.$$

Question 9. $\dfrac{x^2-x+6}{x^3+3x}$.

$$\frac{x^2-x+6}{x^3+3x} = \frac{x^2-x+6}{x(x^2+3)} = \frac{A}{x} + \frac{Bx+C}{x^2+3}.$$

Question 10. $\dfrac{4x}{(x^3+x^2+x+1)^2}$.

$$\begin{aligned}\frac{4x}{(x^3+x^2+x+1)^2} &= \frac{4x}{[(x+1)(x^2+1)]^2} = \frac{4x}{(x+1)^2(x^2+1)^2} \\ &= \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{Cx+D}{x^2+1} + \frac{Ex+F}{(x^2+1)^2}.\end{aligned}$$

INSTRUCTIONS: In questions 11 to 15, evaluate the given integrals, if possible.

Question 11. $\displaystyle\int_0^\infty \frac{1}{\sqrt[4]{1+x}}\,dx.$

$$\int_0^\infty \frac{1}{\sqrt[4]{1+x}}\,dx = \frac{4}{3}\sqrt[4]{(1+x)^3}\Big|_0^\infty = \infty.$$

Question 12. $\displaystyle\int_0^\infty \frac{x\arctan x}{(1+x^2)^2}\,dx.$

Put $z = \arctan x$ so

$$\begin{gathered} dz = \frac{1}{1+x^2}\,dx \\ x = \tan z \\ \frac{1}{1+x^2} = \frac{1}{1+\tan^2 z} = \frac{1}{\sec^2 z}. \end{gathered}$$

Then

$$\begin{aligned}\int_0^\infty \frac{x\arctan x}{(1+x^2)^2}\,dx &= \int_0^{\frac{\pi}{2}} \frac{\tan z}{\sec^2 z} z\,dz = \int_0^{\frac{\pi}{2}} z\sin z\cos z\,dz \\ &= \frac{1}{2}\int_0^{\frac{\pi}{2}} z\sin(2z)\,dz,\end{aligned}$$

where we used $\sin(2z) = 2\sin z\cos z$. Now use integration by parts with

$$\begin{gathered} u = z \Rightarrow du = dz \\ dv = \sin(2z)\,dz \Rightarrow v = -\frac{1}{2}\cos(2z) \end{gathered}$$

so

$$\begin{aligned}\int_0^\infty \frac{x\arctan x}{(1+x^2)^2}\,dx &= \frac{1}{2}\int_0^{\frac{\pi}{2}} z\sin(2z)\,dz \\ &= -\frac{1}{2}\frac{z\cos(2z)}{2}\Big|_0^{\frac{\pi}{2}} + \frac{1}{2}\int_0^{\frac{\pi}{2}} \cos(2z)\,dz \\ &= \frac{\pi}{8} + \frac{1}{8}\sin(2z)\Big|_0^{\frac{\pi}{2}} = \frac{\pi}{8}.\end{aligned}$$

Question 13. $\displaystyle\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx.$

$$\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x\Big|_0^1 = \frac{\pi}{2}.$$

Question 14. $\displaystyle\int_0^{10} \frac{1}{(x-\pi)^7}\,dx.$

Write

$$\int_0^{10} \frac{1}{(x-\pi)^7}\,dx = \int_0^{\pi} \frac{1}{(x-\pi)^7}\,dx + \int_{\pi}^{10} \frac{1}{(x-\pi)^7}\,dx.$$

But

$$\int_0^{\pi} \frac{1}{(x-\pi)^7}\,dx = \lim_{t\to\pi^-} \int_0^{t} \frac{1}{(x-\pi)^7}\,dx = \lim_{t\to\pi^-} -\frac{1}{6}\frac{1}{(x-\pi)^6} = \infty,$$

so the integral diverges.

Question 15. $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{9+x^6}\,dx.$

Put $u = x^3$. Then

$$\int_{-\infty}^{\infty} \frac{x^2}{9+x^6}\,dx = \frac{1}{3}\int_{-\infty}^{\infty} \frac{du}{9+u^2} = \frac{1}{3}\frac{1}{3}\arctan\frac{u}{3}\Big|_{-\infty}^{\infty} = \frac{1}{9}\pi.$$

INSTRUCTIONS: In questions 16 to 18, determine whether the given integral converges.

Question 16. $\displaystyle\int_1^{\infty} \frac{\sin^2 x}{x^2}\,dx.$

Since $\sin^2 x \leq 1$,

$$\int_1^{\infty} \frac{\sin^2 x}{x^2}\,dx \leq \int_1^{\infty} \frac{1}{x^2}\,dx.$$

The integral on the right is a p-integral with $p = 2$, so it converges. Hence $\int_1^{\infty} \frac{\sin^2 x}{x^2}\,dx$ converges as well.

Question 17. $\displaystyle\int_0^{\infty} \frac{e^{x^2}}{e^{3x}+e^{2x}+8}\,dx.$

For $x \geq 0$:

$$e^{2x} \leq e^{3x},$$

hence

$$\int_0^{\infty} \frac{e^{x^2}}{e^{3x}+e^{2x}+8}\,dx \geq \int_0^{\infty} \frac{e^{x^2}}{2e^{3x}+8}\,dx.$$

Since

$$8 \leq 2e^{3x} \Leftrightarrow \frac{1}{3}\ln 4 \leq x,$$

we have

$$\begin{aligned}\int_0^\infty \frac{e^{x^2}}{e^{3x}+e^{2x}+8}\,dx &\geq \int_0^\infty \frac{e^{x^2}}{2e^{3x}+8}\,dx \\ &= \int_0^{\frac{1}{3}\ln 4} \frac{e^{x^2}}{2e^{3x}+8}\,dx + \int_{\frac{1}{3}\ln 4}^\infty \frac{e^{x^2}}{2e^{3x}+8}\,dx \\ &\geq \int_0^{\frac{1}{3}\ln 4} \frac{e^{x^2}}{2e^{3x}+8}\,dx + \int_{\frac{1}{3}\ln 4}^\infty \frac{e^{x^2}}{4e^{3x}}\,dx \\ &= \int_0^{\frac{1}{3}\ln 4} \frac{e^{x^2}}{2e^{3x}+8}\,dx + \frac{1}{4}\int_{\frac{1}{3}\ln 4}^\infty e^{x^2-3x}\,dx.\end{aligned}$$

The last integral on the right hand side obviously diverges to ∞, so $\int_0^\infty \frac{e^{x^2}}{e^{3x}+e^{2x}+8}\,dx$ diverges as well.

Question 18. $\displaystyle\int_0^1 \frac{1}{x\sqrt{x^2+1}}\,dx.$

Notice that $x^2+1 \leq 2$ for $0 \leq x \leq 1$, therefore

$$\int_0^1 \frac{1}{x\sqrt{x^2+1}}\,dx \geq \frac{1}{\sqrt{2}}\int_0^1 \frac{1}{x}\,dx.$$

But

$$\int_0^1 \frac{1}{x}\,dx = \infty,$$

so $\int_0^1 \frac{1}{x\sqrt{x^2+1}}\,dx$ diverges.

INSTRUCTIONS: In questions 19 and 20, set up an integral for the area of the surface obtained by rotating the given curve about the indicated axis. Do not evaluate the integral.

Question 19. $y = \sin^2 x,\ -\pi \leq x \leq \pi,$ about the $x-$axis.

$$\begin{aligned}S &= \int_{-\pi}^{\pi} 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \\ &= \int_{-\pi}^{\pi} 2\pi \sin^2 x\sqrt{1+(2\sin x\cos x)^2}\,dx\end{aligned}$$

Question 20. $y = e^x,\ 1 \leq x \leq 2,$ about $x = -1.$

Put $u = x+1$, so the curve becomes

$$y = e^{u-1},\ 2 \leq u \leq 3,$$

rotated about the u-axis. Then

$$\begin{aligned} S &= \int_2^3 2\pi u \sqrt{1 + \left(\frac{dy}{du}\right)^2}\, du \\ &= \int_2^3 2\pi u \sqrt{1 + e^{2u-2}}\, du. \end{aligned}$$