MAT 155B - FALL 12 - EXAMPLES SECTION 9.3

PROBLEMS

Question 1. Solve the differential equation:

$$(a)\ y' = \frac{te^t}{y\sqrt{1+y^2}} \qquad (b)\ \frac{du}{dt} = 2 + 2u + t + tu$$

Question 2. Find the solution of the differential equation that satisfies the given initial conditions.

$$(a)\ y' = \frac{x}{y},\ y(0) = -3 \qquad (b)\ x\ln x = y(1+\sqrt{3+y^2})y',\ y(1) = 1$$

Question 3. A vat with 500 gallons of beer contains 4% of alcohol (by volume). Beer with 6% alcohol is pumped into the vat at a rate 5 gal/min and the mixture is pumped out at the same rate. What is the percentage of alcohol after an hour?

SOLUTIONS

1a. Write $y' = \frac{dy}{dx}$ and

$$y\sqrt{1+y^2}\,dy = te^t\,dt \Rightarrow \int y\sqrt{1+y^2}\,dy = \int te^t\,dt$$

For the y-integral, make the substitution $u = 1+y^2$, so $du = 2ydy$ and

$$\int y\sqrt{1+y^2}\,dy = \frac{1}{2}\int \sqrt{u}\,du = \frac{1}{3}u^{\frac{3}{2}} = \frac{1}{3}(1+y^2)^{\frac{3}{2}}$$

For the t integral, recall the formula for integration by parts

$$\int u\,dv = uv - \int v\,du$$

and choose $u = t$, $dv = e^t\,dt$, so that

$$du = dt$$
$$v = \int dv = \int e^t\,dt = e^t$$

and therefore

$$\int te^t\,dt = te^t - \int e^t\,dt = e^t(t-1) + C$$

Putting the y and t integrals together,

$$\frac{1}{3}(1+y^2)^{\frac{3}{2}} = e^t(t-1) + C$$

Solving for y

$$y = \pm\sqrt{(3e^t(t-1)+C)^{\frac{2}{3}} - 1}$$

1b. Write

$$\frac{du}{dt} = 2 + 2u + t + tu = (1+u)(2+t)$$

Then

$$\int \frac{du}{1+u} = \int (2+t)\,dt = 2\int dt + \int t\,dt$$

Hence

$$\ln|1+u| = 2t + \frac{t^2}{2} + C$$

where we used $\int \frac{du}{1+u} = \ln|1+u|$. Take ln in both sides to solve for u

$$|1+u| = e^C e^{2t+\frac{t^2}{2}} \Rightarrow u = \pm e^C e^{2t+\frac{t^2}{2}} - 1 = Ae^{2t+\frac{t^2}{2}} - 1$$

for some undetermined constant A, i.e., as in class, we relabeled the undermined constant.

2a. Write

$$y\,dy = x\,dx \Rightarrow \int y\,dy = \int x\,dx \Rightarrow y^2 = x^2 + C \Rightarrow y = \pm\sqrt{x^2+C}$$

Since $y(0) = -3$, plugging in we find:

$$-3 = \pm\sqrt{0+C} \Rightarrow C = 9$$

and the solution is the one with the negative sign in order to obtain -3. Hence

$$y = -\sqrt{x^2+9}$$

2b. Write

$$x\ln x = y(1+\sqrt{3+y^2})\frac{dy}{dx} \Rightarrow x\ln x\,dx = y(1+\sqrt{3+y^2})\,dy$$

Integrating

$$\int x\ln x\,dx = \int y(1+\sqrt{3+y^2})\,dy = \int y\,dt + \int y\sqrt{3+y^2}\,dy$$

For the t integral, use integration by parts with $u = \ln x$, $dv = x\,dx$, so that

$$du = \frac{1}{x}$$

$$v = \int dv = \int x\,dx = \frac{1}{2}x^2$$

and therefore

$$\int x\ln x\,dx = \ln x\frac{1}{2}x^2 - \int \frac{1}{2}x^2\frac{1}{x}\,dx = \frac{1}{2}x^2\ln x - \frac{1}{4}x^2$$

The first y integral gives $\frac{1}{2}y^2$. For the second one, make the substitution $u = 3+y^2$, so that $du = 2ydy$ and hence

$$\int y\sqrt{3+y^2}\,dy = \frac{1}{2}\int \sqrt{u}\,du = \frac{1}{3}u^{\frac{3}{2}} = \frac{1}{3}(3+y^2)^{\frac{3}{2}}$$

Therefore

$$\int y(1+\sqrt{3+y^2})\,dy = \int y\,dy + \int \sqrt{3+y^2}\,dy = \frac{1}{2}y^2 + \frac{1}{3}(3+y^2)^{\frac{3}{2}} + C$$

Hence

$$\frac{1}{2}x^2\ln x - \frac{1}{4}x^2 = \frac{1}{2}y^2 + \frac{1}{3}(3+y^2)^{\frac{3}{2}} + C$$

To find the value of the constant C we use $y(1) = 1$ to get

$$\frac{1}{2}1^2\underbrace{\ln 1}_{=0} - \frac{1}{4}1^2 = \frac{1}{2}1^2 + \frac{1}{3}(3+1^2)^{\frac{3}{2}} + C \Rightarrow C = -\frac{41}{12}$$

And hence the solution is

$$\frac{1}{2}y^2 + \frac{1}{3}(3+y^2)^{\frac{3}{2}} - \frac{41}{12} = \frac{1}{2}x^2\ln x - \frac{1}{4}x^2$$

Notice that this is an implicit solution.

3. Let's first find an equation for the amount of alcohol. Denote the amount of alcohol at time t by $q(t)$. One needs to find an equation for $\frac{dq}{dt}$, which will be given by

$$\frac{dq}{dt} = \text{(rate in)} - \text{(rate out)}$$

The first thing to notice is that the volume of the vat remains constant and equals to 500 gallons. Notice also that $\frac{dq}{dt}$ is measured in gallons per minute, so are (rate in) and (rate out). To find rate in, notice that 6% of 5 is 0.3, hence if a mixture with 6% of alcohol enters per minute we get

$$\text{(rate in)} = 0.3\ gal/min$$

The rate out is going to be the concentration of alcohol times the rate at which the mixture flows out of the vat. The concentration of alcohol is simply the amount of alcohol, $q(t)$, divide by the total volume, 500 gallons. Hence

$$\text{(rate out)} = \frac{q(t)}{500} \times 5 = \frac{q(t)}{100}\ gal/min$$

Hence

$$\frac{dq}{dt} = 0.3 - \frac{q}{100}$$

Or

$$\frac{dq}{30-q} = \frac{1}{100}\,dt \Rightarrow \int \frac{dq}{30-q} = \int \frac{1}{100}\,dt$$

Computing the integrals we find

$$-\ln|30-q| = 0.01t + C \Rightarrow q = Ae^{-0.01t} + 30$$

To find A we need to use the initial condition. At time zero there is 4% of alcohol, which corresponds to

$$4\% \text{ of } 500\,gal = 20\,gal$$

Hence $q(0) = 20$ and

$$20 = Ae^0 + 30 \Rightarrow A = -10$$

Therefore

$$q = -10e^{-0.01t} + 30$$

Plugging $t = 60$ minutes gives $q(60) = 24.51$, or 4.9% of alcohol.

URL: http://www.disconzi.net/Teaching/MAT155B-Fall-12/MAT155B-Fall-12.html