MAT 155B - FALL 12 - EXAMPLES SECTION 7.3 AND 7.4

Question 1. Compute

$$\int \frac{dx}{x^2\sqrt{x^2+a^2}}.$$

Question 2. Compute

$$\int_2^3 \frac{4}{x^2-1}dx.$$

Solutions.

1. Use the substitution $x = a\tan\theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. From $x = a\tan\theta$ we get $dx = a\sec^2\theta d\theta$ and

$$\sqrt{x^2+a^2} = \sqrt{a^2+a^2\tan^2\theta} = \sqrt{a^2(1+\tan^2\theta)} = a\sqrt{1+\tan^2\theta}. \tag{1}$$

Recalling that $1+\tan^2\theta = \sec^2\theta$ we have

$$\int \frac{dx}{x^2\sqrt{x^2+a^2}} = \int \frac{a\sec^2\theta d\theta}{a^2\tan^2\theta a\sec\theta} = \frac{1}{a^2}\int \frac{\sec\theta}{\tan^2\theta}d\theta = \frac{1}{a^2}\int \frac{\cos\theta}{\sin^2\theta}d\theta,$$

where in the last step we used $\sec\theta = \frac{1}{\cos\theta}$. Now do the substitution $u = \sin\theta$. Then $du = \cos\theta d\theta$ and we obtain

$$\frac{1}{a^2}\int \frac{\cos\theta}{\sin^2\theta}d\theta = \frac{1}{a^2}\int \frac{du}{u^2} = -\frac{1}{a^2}\frac{1}{u} + C,$$

where C is a constant. Now we need to return to the variable θ and then to x. Since $u = \sin\theta$ we have

$$-\frac{1}{a^2}\frac{1}{u} + C = -\frac{1}{a^2}\frac{1}{\sin\theta} + C.$$

To write the answer in terms of x, first write the substitution $x = a\tan\theta$ as $\tan\theta = \frac{x}{a}$ and then notice that this gives $\sin\theta = \frac{x}{\sqrt{x^2+a^2}}$; see figure below.

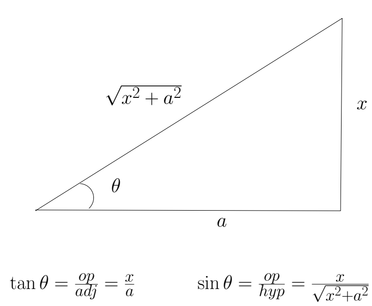

FIGURE 1. Geometrical interpretation of the variable θ in terms of x.

Then

$$-\frac{1}{a^2}\frac{1}{\sin\theta}+C=-\frac{1}{a^2}\frac{1}{x/\sqrt{x^2+a^2}}+C=-\frac{1}{a^2}\frac{\sqrt{x^2+a^2}}{x}+C,$$

so

$$\int\frac{dx}{x^2\sqrt{x^2+a^2}}=-\frac{1}{a^2}\frac{\sqrt{x^2+a^2}}{x}+C.$$

2. Write $\frac{4}{x^2-1}$ as $\frac{4}{(x+1)(x-1)}$. Let us write this expression as a sum, i.e.,

$$\frac{4}{(x+1)(x-1)}=\frac{A}{x+1}+\frac{B}{x-1}=\frac{Ax-A+Bx+B}{(x+1)(x-1)}=\frac{(A+B)x+(B-A)}{(x+1)(x-1)}.$$

So, in order to have the equality

$$\frac{4}{(x+1)(x-1)}=\frac{(A+B)x+(B-A)}{(x+1)(x-1)}$$

the numerators in both sides must be equal. Since in the numerator of the left hand side we do not have x, we obtain:

$$\begin{cases} A+B=0, \\ B-A=4. \end{cases}$$

Solving for A and B, we find $A=-2$ and $B=2$. Hence,

$$\int_2^3\frac{4}{x^2-1}dx=\int_2^3\frac{2}{x-1}dx-\int_2^3\frac{2}{x+1}dx=$$
$$2\ln(x-1)\Big|_2^3-2\ln(x+1)\Big|_2^3=2\ln 2-2\ln 4+2\ln 3=2\ln 3-\ln 4$$

where we used $2\ln 2=\ln 2^2=\ln 4$.