MAT 155B - FALL 12 - EXAMPLES SECTION 7.3

Question 1. Use integration to show that

$$\sinh^{-1}\frac{x}{a} = \ln(x+\sqrt{x^2+a^2}) - \ln a,$$

for $a > 0$.

Question 2. Compute the integrals

$$(a) \int \frac{1}{\sqrt{5-4x-2x^2}}\,dx.$$

$$(b) \int_5^{10} \frac{\sqrt{x^2-25}}{x}\,dx.$$

Solutions.

1. Consider

$$\int \frac{dx}{\sqrt{a^2+x^2}}. \tag{1}$$

We shall compute this integral in two different ways.

Let $x = a\tan\theta$, with $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Then

$$dx = a\sec^2\theta\,d\theta,$$

and

$$\sqrt{x^2+a^2} = \sqrt{a^2\tan^2\theta + a^2} = \sqrt{a^2(\tan^2\theta+1)} = \sqrt{a^2\sec^2\theta} = a\sec\theta.$$

Therefore

$$\int \frac{dx}{\sqrt{a^2+x^2}} = \int \frac{a\sec^2\theta}{a\sec\theta}\,d\theta = \int \sec\theta\,d\theta.$$

To compute the last integral, multiply and divide the integrand by $\sec\theta+\tan\theta$,

$$\int \sec\theta\,d\theta = \int \sec\theta\frac{\sec\theta+\tan\theta}{\sec\theta+\tan\theta}\,d\theta = \int \frac{\sec^2\theta+\sec\theta\tan\theta}{\sec\theta+\tan\theta}\,d\theta.$$

Now make the u-substitution $u = \sec\theta+\tan\theta$, so that $du = (\sec\theta\tan\theta+\sec^2\theta)d\theta$ and

$$\int \frac{\sec^2\theta+\sec\theta\tan\theta}{\sec\theta+\tan\theta}\,d\theta = \int \frac{du}{u} = \ln|u| + C = \ln|\sec\theta+\tan\theta| + C.$$

But $\tan\theta = \frac{x}{a}$, whereas $\sec\theta = \frac{\sqrt{a^2+x^2}}{a}$, so

$$\int \frac{dx}{\sqrt{a^2+x^2}} = \ln\left|\frac{\sqrt{a^2+x^2}}{a} + \frac{x}{a}\right| = \ln(x+\sqrt{a^2+x^2}) - \ln a + C, \tag{2}$$

where the absolute value has been removed from the first ln because $\sqrt{a^2+x^2}+x > 0$.

Now let us compute integral (1) in a different way. Let $x = a\sinh\xi$, so that

$$dx = a\cosh\xi\,d\xi,$$

and
$$\sqrt{x^2+a^2} = a\cosh\xi,$$
where we have used $\sinh^2\xi - \cosh^2\xi = 1$. Hence
$$\int \frac{dx}{\sqrt{a^2+x^2}} = \int \frac{a\cosh\xi}{a\cosh\xi}\,d\xi = \int d\xi = \xi + D,$$
where D is an arbitrary constant. But $\xi = \sinh^{-1}\frac{x}{a}$ and so we conclude
$$\int \frac{dx}{\sqrt{x^2+a^2}} = \sinh^{-1}\frac{x}{a} + D. \tag{3}$$
Setting (2) equal to (3) yields
$$\ln(x+\sqrt{a^2+x^2}) - \ln|a| + C = \sinh^{-1}\frac{x}{a} + D.$$
Evaluating at zero and using that $\sinh(0) = 0$ we conclude that $C = D$, so
$$\sinh^{-1}\frac{x}{a} = \ln(x+\sqrt{a^2+x^2}) - \ln|a|,$$
as desired.

2a. Write
$$\begin{aligned}5-4x-2x^2 &= -2(x+1)^2+7 = 7-2(x+1)^2 = 7-\left(\sqrt{2}(x+1)\right)^2\\ &= 7\left[1-\left(\frac{\sqrt{2}(x+1)}{\sqrt{7}}\right)^2\right].\end{aligned}$$
Therefore
$$\int \frac{1}{\sqrt{5-4x-2x^2}}\,dx = \int \frac{dx}{\sqrt{7}\sqrt{1-\left(\frac{\sqrt{2}(x+1)}{\sqrt{7}}\right)^2}}.$$
Making the substitution $u = \frac{\sqrt{2}(x+1)}{\sqrt{7}}$ gives
$$\begin{aligned}\int \frac{1}{\sqrt{5-4x-2x^2}}\,dx &= \frac{1}{\sqrt{2}}\int \frac{du}{\sqrt{1-u^2}} = \frac{1}{\sqrt{2}}\arcsin u + C\\ &= \frac{1}{\sqrt{2}}\arcsin\frac{\sqrt{2}(x+1)}{\sqrt{7}} + C.\end{aligned}$$

2b. There are two approaches.
Approach 1. Do the indefinite integral, plugging the limits later.
Making the substitution $x = 5\sec\theta$, $0 \le \theta < \frac{\pi}{2}$ gives
$$\begin{aligned}\int \frac{\sqrt{x^2-25}}{x}\,dx &= \int \frac{25\tan\theta\sec\theta\tan\theta}{\sec\theta}\,d\theta = 5\int \tan^2\theta\,d\theta = 5\int(\sec^2\theta-1)d\theta\\ &= 5\tan\theta - 5\theta + C = \sqrt{x^2-25} - 5\arctan\frac{\sqrt{x^2-25}}{5} + C.\end{aligned}$$
Plugging the limits:
$$\int_5^{10} \frac{\sqrt{x^2-25}}{x}\,dx = \sqrt{x^2-25}\Big|_5^{10} - 5\arctan\frac{\sqrt{x^2-25}}{5}\Big|_5^{10} = 5\sqrt{3} - \frac{5\pi}{3}.$$

Approach 2. Change the limits of integrations.
Do the same substitution as before, i.e., $x = 5\sec\theta$. Then

$$x = 5 \Rightarrow \sec\theta = 1 \Rightarrow \theta = 0$$
$$x = 10 \Rightarrow \sec\theta = 2 \Rightarrow \theta = \frac{\pi}{3}.$$

So

$$\int_0^{10} \frac{\sqrt{x^2-25}}{x}\,dx = 5\int_0^{\frac{\pi}{3}} \tan^2\theta\,d\theta$$
$$= 5\tan\theta\Big|_0^{\frac{\pi}{3}} - 5\theta\Big|_0^{\frac{\pi}{3}} = 5\sqrt{3} - \frac{5\pi}{3}.$$

URL: `http://www.disconzi.net/Teaching/MAT155B-Fall-12/MAT155B-Fall-12.html`