MAT 155B - FALL 12 - EXAMPLES SECTION 6.5

PROBLEMS

Question 1. Strontium-90, ^{90}Sr, has a half-life of 28 years[1]. Find a formula for the mass remaining after t years. How long does it take for a sample to decay to 8% of its initial amount?

Question 2. Consider again the substance Strontium-90. A sample has a mass of 50 mg.

(a) Find a formula for the mass remaining after t years.

(b) Find the mass remaining after 40 years.

(c) How long does it take for the sample to decay to a mass of 2 mg?

Question 3. Suppose a population $P(t)$ is modeled by

$$\frac{dP}{dt} = 0.025P - 0.0005P^2,$$

known as logistic equation. If the initial population is 200, find a formula for the population at time t.

SOLUTIONS

1. Recall that for half-life problems we always use the differential equation

$$\frac{dy}{dt} = ky.$$

We saw in class that the solution is given by $y = y_0 e^{kt}$, where y_0 is the quantity at time zero. The value of k depends on the particular substance/process under investigation. In other words, the value of k will vary from problem to problem, and in most exercises the first thing to do is to find the value of k. For this we use the half life:

$$y(28) = \frac{1}{2}y_0 = y_0 e^{k28} \Rightarrow k = \frac{1}{28}\ln\frac{1}{2} = -0.025.$$

Whenever using the half-life, we plug in the value of the half-life for t and $\frac{y_0}{2}$ for $y(t)$, since by definition the half-life is the time required for half of the substance to decay.

Hence we obtain

$$y(t) = y_0 e^{-0.025t}.$$

For the second part of the question, notice that 8% of the initial amount will be $0.08y_0$, hence

$$0.08y_0 = y_0 e^{-0.025t} \Rightarrow t = -\frac{1}{0.025}\ln(0.08) = 101 \; years.$$

[1] According to Wikipedia, "Natural strontium is nonradioactive and nontoxic, but ^{90}Sr is a radioactivity hazard". It has "extensive use in medicine and industry, as a radioactive source for thickness gauges and for superficial radiotherapy of some cancers".

2a. In problem 1 we found $y(t) = y_0 e^{-0.025t}$. So all we have to do is to set $y_0 = 50$:

$$y(t) = 50e^{-0.025t}.$$

2b. After 40 years means $t = 40$, so

$$y(40) = 50e^{-0.025 \cdot 40} = 18.40\ mg.$$

2c. Now we set $y(t) = 2\ mg$ and solve for t

$$y(t) = 2 = 50e^{-0.025t} \Rightarrow t = -\frac{1}{0.025} \ln \frac{2}{50} = 128.75\ years.$$

3. Write the equations as

$$\frac{dP}{0.025P - 0.0005P^2} = dt \Rightarrow \int \frac{dP}{0.025P - 0.0005P^2} = \int dt.$$

Factor P on the denominator of the left hand side and write

$$(\star) \qquad \frac{1}{P(0.025 - 0.0005P)} = \frac{A}{P} + \frac{B}{0.025 - 0.0005P}.$$

Let's find A and B. For this, take the least common multiple of the right hand side (notice that this essentially boils down to multiplying the bottom and "cross multiplying" to top) to get

$$\frac{A}{P} + \frac{B}{0.025 - 0.0005P} = \frac{A(0.025 - 0.0005P) + BP}{P(0.025 - 0.0005P)}.$$

On the other hand, this has to be equal to $(\star)$, hence

$$\frac{1}{P(0.025 - 0.0005P)} = \frac{A(0.025 - 0.0005P) + BP}{P(0.025 - 0.0005P)}.$$

Canceling the denominators on both sides gives

$$1 = A(0.025 - 0.0005P) + BP.$$

To find A and B, we plug in for P first the value that makes B disappear, i.e., $P = 0$, and then the value that makes A disappear, i.e., $0.025 - 0.0005P = 0 \Rightarrow P = \frac{0.025}{0.0005} = 50$. Plugging $P = 0$ gives

$$1 = A(0.025 - 0.0005 \cdot 0) + B \cdot 0 \Rightarrow A = \frac{1}{0.025} = 40.$$

Plugging $P = 50$ gives

$$1 = A \cdot 0 + B \cdot 50 \Rightarrow B = \frac{1}{50} = 0.02.$$

Notice that we don't need to compute the number multiplying A, we know that it will be zero, since that's how we found the value 50 in the first place.

Hence $(\star)$ becomes

$$\frac{1}{P(0.025 - 0.0005P)} = \frac{40}{P} + \frac{0.02}{0.025 - 0.0005P}.$$

Now we integrate

$$\int \frac{dP}{P(0.025 - 0.0005P)} = 40 \int \frac{dP}{P} + 0.02 \int \frac{dP}{0.025 - 0.0005P}.$$

Performing the integrals gives

$$\int \frac{dP}{P(0.025 - 0.0005P)} = 40 \ln |P| - \frac{0.02}{0.0005} \ln |0.025 - 0.0005P|.$$

Simplify and use the property $\ln a - \ln b = \ln \frac{a}{b}$ to get

$$40 \ln |P| - 40 \ln |0.025 - 0.0005P| = 40 \ln \frac{|P|}{|0.025 - 0.0005P|}.$$

Therefore

$$\int \frac{dP}{0.025P - 0.0005P^2} = \int dt.$$

becomes

$$40 \ln \frac{|P|}{|0.025 - 0.0005P|} = t + C.$$

Now we need to solve for P and find the value of the constant of integration. Divide by 40

$$\ln \frac{|P|}{|0.025 - 0.0005P|} = \frac{t}{40} + \frac{C}{40},$$

and exponentiate

$$\frac{|P|}{|0.025 - 0.0005P|} = e^{\frac{t}{40} + \frac{C}{40}} = e^{\frac{C}{40}} e^{\frac{t}{40}} = K e^{\frac{t}{40}},$$

where we labeled the constant $e^{\frac{C}{40}}$ by K. Removing the absolute values gives a plus or minus sign:

$$\frac{P}{0.025 - 0.0005P} = \pm K e^{\frac{t}{40}} = A e^{\frac{t}{40}},$$

where again we relabeled constants with $\pm K = A$. Now use the initial condition $P(0) = 200$ to get

$$\frac{200}{0.025 - 0.0005 \cdot 200} = A e^{\frac{0}{40}} = A \Rightarrow A = -2666.67.$$

So

$$\frac{P}{0.025 - 0.0005P} = -2666.67 e^{\frac{t}{40}}.$$

Now let's solve for P. Multiply by $0.025 - 0.0005P$,

$$P = -2666.67 e^{\frac{t}{40}} (0.025 - 0.0005P) = -66.67 e^{\frac{t}{40}} + 1.33 e^{\frac{t}{40}} P,$$

so

$$P - 1.33 e^{\frac{t}{40}} P = (1 - 1.33 e^{\frac{t}{40}}) P = -66.67 e^{\frac{t}{40}},$$

and therefore

$$P = \frac{-66.67 e^{\frac{t}{40}}}{1 - 1.33 e^{\frac{t}{40}}}.$$

Although this is the answer, let's simplify it. Divide the top and bottom by $-e^{\frac{t}{40}}$:

$$P = \frac{66.67}{-e^{-\frac{t}{40}} + 1.33},$$

and divide top and bottom further by 1.33 to get

$$P(t) = \frac{50.12}{1 - 0.75 e^{-\frac{t}{40}}}.$$

Remark 1. The trick of writing the original equation as ($\star$), and then finding A and B will be studied in more detail in class, and it is known as *partial fractions*.

URL: `http://www.disconzi.net/Teaching/MAT155B-Fall-12/MAT155B-Fall-12.html`