MAT 155B - FALL 12 - EXAMPLES SECTION 6.3*

The *error function* $\operatorname{erf}(x)$ **is defined as**

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt. \tag{1}$$

Show that

$$\int_a^b e^{-x^2} dx = \frac{\sqrt{\pi}}{2} \left(\operatorname{erf}(b) - \operatorname{erf}(a) \right).$$

Solution. Taking the derivative of (1) and using the fundamental theorem of calculus gives

$$\frac{d}{dx} \operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} e^{-x^2}.$$

Therefore

$$\int_a^b e^{-x^2} dx = \int_a^b \frac{\sqrt{\pi}}{2} \frac{d}{dx} \operatorname{erf}(x) dx = \frac{\sqrt{\pi}}{2} \left(\operatorname{erf}(b) - \operatorname{erf}(a) \right),$$

as desired.

The error function has important applications in the statistical analysis of data. For instance, consider a series of measurements described by a normal distribution with standard deviation σ and expected value 0. Then the probability that the a single measurement lies between, say, $-a$ and a, is given by

$$\operatorname{erf}\left(\frac{a}{\sqrt{2}\sigma} \right).$$

Notice that the probability that a measurement has any value, i.e., lies between $-\infty$ and ∞, has to be equal to one. Therefore, assuming $\sigma = 1$ for simplicity, we conclude that

$$\operatorname{erf}(+\infty) = \lim_{a \to +\infty} \operatorname{erf}\left(\frac{a}{\sqrt{2}} \right) = 1.$$

But this then means, using (1), that

$$\int_0^\infty e^{-t^2} dt = \frac{\sqrt{\pi}}{2}. \tag{2}$$

Formula (2) can be shown by direct computation, without relying in the aforementioned interpretation of the error function.

A natural question which arises in the present context is the following. Why do we need to define erf as the integral of another function? Why not simply *compute* the integral $\int_0^x e^{-t^2} dt$ and define the error function as the resulting expression times $\frac{2}{\sqrt{\pi}}$?

The answer is that $\int_0^x e^{-t^2} dt$ *cannot be computed*, in the sense that one cannot express its value in terms of elementary functions. In other words, although $\operatorname{erf}(x)$ is a well defined function, it is impossible to find a formula of the form

$$\int_0^x e^{-t^2} dt = \text{ some closed expression involving } e^{x^2},\ e^x \text{ and powers of } x.$$

But one can find a formula for $\int_0^x e^{-t^2} dt$ using power series! We will return to this point when we study chapter 11.

URL: `http://www.disconzi.net/Teaching/MAT155B-Fall-12/MAT155B-Fall-12.html`