MAT 155B - FALL 12 - EXAMPLES SECTION 6.1

Recall that we saw in class that a function f with domain A and range B is *one-to-one* (aka *injective*, *into* or *invertible*) if it never takes on the same value twice, i.e.,

$$f(x_1) = f(x_2) \Leftrightarrow x_1 = x_2.$$

In this case the inverse of f, denoted f^{-1}, has domain B and range A, and and it is defined by

$$f^{-1}(y) = x \Leftrightarrow y = f(x).$$

Moreover, it holds that

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}, \tag{1}$$

provided that f is differentiable and $f'(f^{-1}(x)) \neq 0$.

Example. Let $f(x) = \frac{1}{x^3-9}$. (a) Show that f is one-to-one; (b) find $f^{-1}(x)$, and (c) state its domain and range; (d) calculate $(f^{-1})'(1)$.

Solution. (a) Suppose that f takes on the same value on two points x_1 and x_2, i.e., $f(x_1) = f(x_2)$. We need to show that $x_1 = x_2$. But

$$f(x_1) = f(x_2) \Rightarrow \frac{1}{x_1^3 - 9} = \frac{1}{x_2^3 - 9} \Rightarrow x_1^3 - 9 = x_2^3 - 9 \Rightarrow x_1^3 = x_2^3 \Rightarrow x_1 = x_2,$$

where in the last step we used that x^3 has inverse $\sqrt[3]{x}$ (notice that if we had, say, $x_1^2 = x_2^2$, we could not conclude that $x_1 = x_2$ since x^2 is not a one-to-one function).

(b) To find the inverse, write

$$y = \frac{1}{x^3 - 9},$$

and solve for x:

$$(x^3 - 9)y = 1 \Rightarrow x^3 - 9 = \frac{1}{y} \Rightarrow x^3 = \frac{1}{y} + 9 \Rightarrow x = \sqrt[3]{\frac{9y+1}{y}},$$

hence

$$f^{-1}(x) = \sqrt[3]{\frac{9x+1}{x}}. \tag{2}$$

Remark 1. One way of showing that a function is one-to-one is to find its inverse, i.e., show that f^{-1} exists. Hence, in practice, you could solve parts (a) and (b) simultaneously by just doing part (b).

(c) The function $\frac{1}{x^3}$ has domain and range all real numbers but zero. $\frac{1}{x^3-9}$ simply shifts the vertical asymptote from zero to $x = \sqrt[3]{9}$ (the point where the denominator vanishes). Hence $f(x)$ has domain $\mathbb{R} - \{\sqrt[3]{9}\}$ and range $\mathbb{R} - \{0\}$. Recalling that

$$\text{domain of } f = \text{ range of } f^{-1}$$
$$\text{range of } f = \text{ domain of } f^{-1},$$

we conclude that the domain of f^{-1} is $\mathbb{R} - \{0\}$ and its range is $\mathbb{R} - \{\sqrt[3]{9}\}$.

(d) We want to use formula (1). First we compute

$$f'(x) = \frac{d}{dx}\frac{1}{x^3-9} = -\frac{3x^2}{(x^3-9)^2},$$

where the chain rule has been employed. Now compute $f^{-1}(1)$ by plugging 1 in formula (2): $f^{-1}(1) = \sqrt[3]{10}$. Hence

$$(f^{-1})'(1) = \frac{1}{f'(f^{-1}(1))} = -\left.\frac{(x^3-9)^2}{3x^2}\right|_{x=\sqrt[3]{10}} = -\frac{1}{3\cdot 10^{\frac{2}{3}}}.$$

URL: http://www.disconzi.net/Teaching/MAT155B-Fall-12/MAT155B-Fall-12.html